Notes from 2013 La Llagonne Summer School on Donaldson Hypersurfaces

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Contents

1 Introduction - Patrick Massot

Theorem 1.1. *(Donaldson '96)*
If (V, ω) is a closed symplectic manifold with $\frac{1}{2\pi}[\omega] \in H^2(V, \mathbb{Z})$, then for $k >> 1$ there exists Σ symplectic of codimension 2 such that $[\Sigma] = PD(\frac{k}{2\pi}[w])$.

Here $\frac{1}{2\pi}[\omega] \in H^2(V, \mathbb{Z})$ means $\forall s \in H_2(V, \mathbb{Z})$, $\frac{1}{2\pi}\int_S \omega \in \mathbb{Z}$. This means that $\frac{k}{2\pi}\int_S \omega = \Sigma \cdot S$.

Remark 1.2.

- *Recall that Gromov's h principle builds symplectic submanifolds which are either* open *or of codimension ≥ 4.*
- *If V is Kahler, then Kodaira gives an embedding $V \hookrightarrow \mathbb{CP}^N$ and $\Sigma = V \cap (\textit{hypersurface in } \mathbb{CP}^N)$.*
- *Any holomorphic curve in V has nonzero symplectic area, hence has to intersect Σ. This is the basis of Cieliebak-Mohnke (see Chris Wendl's talk).*
- *In dimension 4, the theorem can be used to construct holmorphic curves. Some applications: studying H^* of Hilbert schemes, and showing that $M^3 \times S^1$ symplectic implies that M fibers over S^1.*

Theorem 1.3. *(Biran, Giroux)*
There is a D^2-bundle over Σ whose complement is isotropic.

Some related notions:

- The complement of Σ is Weinstein.
- One can fully (w.r.t. volume) fill V by an ellipsoid.
- Biran, Cieliebak: symplectic or Lagrangian embeddings
- Evans: Nijenhuis energy
- Symplectic isotopies and symplectic mapping class groups

Remark 1.4.

- *In the theorem, $\Sigma = s^{-1}(0)$, where $s : V \to L^k$ is a section (here L^k the is k-fold tensor product of L with itself).*

- *Donaldson also constructs pairs* s_1, s_2 *such that* $B = (s_1, s_2)^{-1}(0,0)$ *is a codimension* 4 *symplectic submanifold, and*

$$V \setminus B \to \mathbb{CP}^1, \quad x \mapsto [s_1(x) : s_2(x)]$$

  is a Lefshetz pencil.

- *After blowing up* B, *we get a Lefshetz fibration* $\tilde{V} \to \mathbb{CP}^1$.

- *Auroux showed that with a third section in dimension* 4, V^4 *becomes a branched cover of* $\mathbb{CP}^2$.

- *There is also a contact version: Ibort-Martinez-Presas show how to construct contact submanifolds.*

- *Giroux-Mohsen show how to construct open book decompositions.*

- *Casals-Pancholi-Presas construct contact structures in dimension* 5.

- *There are applications to rational convexity.*

There are also some connections with fields:

- *Existence of Kahler-Einstein metrics on Fano manifolds*

- *Statistics in real algebraic geometry*

- *Yomdin: dynamical systems, entropy*

1.1 Line Bundles

Let V be a smooth manifold, $V = \cup_{i=1}^{N} U_i$. Assume each U_i, $U_{ij} := U_i \cap U_j$, and $U_{ijk} := U_i \cap U_j \cap U_k$ is contractible. Let $\pi : L \to V$ be a line smooth. We have trivializations:

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[rr, "\phi_i"] \arrow[dr] & & U_i \times \mathbb{C} \arrow[dl] \\
& U_i, &
\end{tikzcd}$$

with

$$\begin{aligned}
&\phi_i \circ \phi_j^{-1} : U_{ij} \times \mathbb{C} \to U_{ij} \times \mathbb{C}, \quad x \mapsto (x, g_{ij}(x)) \\
&g_{ij} : U_{ij} \to \mathbb{C}^*,
\end{aligned}$$

satisfying the “cocycle condtion”:

$$g_{ij}g_{ji} = 1$$
$$g_{ij}g_{jk} = g_{ik}.$$

Conversely, we can use the g_{ij}'s (satisfying the cocycle condition) to build the line bundle L.

If $g_{ij}(x) \in U(1)$, we get a Hermitian structure. If $L \to V$ and $L' \to V$ are two line bundles with transition functions $g_{ij}(x)$ and $g'_{ij}(x)$, the transitions functions for the line bundle $L \otimes L' \to V$ are given by $g_{ij}(x)g'_{ij}(x)$. Note that for s a section of L, we get locally $s_i : U_i \to \mathbb{C}$.

Example 1.5. *Let $V = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, with $U_0 = \mathbb{C}$, $U_1 = \mathbb{CP}^1 \setminus \{0\}$. Then the transition function $g_{01}(z) = z^n$ builds the line bundle $O(n) \to \mathbb{CP}^1$. One can check that holomorphic sections of $O(n)$ correspond to degree $\leq n$ polynomials, and that for such a section s we have $s^{-1}(0) = n$ points.*

1.2 Connections

Recall that a connection ∇ is given by $\nabla : \Gamma(L) \to \Gamma(T^*V \otimes L)$ such that

$$\nabla(fs) = df \otimes s + f\nabla s.$$

We say that ∇ is *Hermitian* if

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle.$$

Recall that over U_i ∇ can be written as

$$\nabla = d + A_i,$$

where A_i is a complex valued 1-form on U_i. If ∇ is Hermitian, then A_i will be purely imaginary.

One can compute that on U_{ij}, $A_i = A_i + g_{ij}^{-1}dg_{ij}$, where $g_{ij}^{-1}dg_{ij}$ is closed, and hence $dA_i = dA_j$. This means we get a well-defined curvature 2-form $F := dA_i$ on V.

Theorem 1.6. *(Chern-Weil)*

Let $F = -i\omega$ be the curvature of a Hermitian line bundle $L \to V$, and let s be a section, transverse to the zero section, with $\Sigma = s^{-1}(0)$. Then for Z any 2-cycle transverse to Σ, we have

$$\frac{1}{2\pi}\int_Z \omega = Z \cdot \Sigma.$$

Note here that $\frac{1}{2\pi}[\omega] \in H^2(V, \mathbb{Z})$.

Proof. Fix $\epsilon > 0$, and set

$$V_\epsilon = \{|s| \geq \epsilon\}$$
$$Z_\epsilon = Z \cap V_\epsilon.$$

On V_ϵ, $\frac{\nabla s}{s}$ is a well-defined complex valued 1-form satisfying

$$d\left(\frac{\nabla s}{s}\right) = -i\omega$$

(since locally we have $\nabla s_i = ds_i + A_i s_i$, hence $d\left(\frac{\nabla s_i}{s_i}\right) = d\left(\frac{ds_i}{s_i}\right) + dA_i = -i\omega$). Then we have

$$\begin{aligned}\int_Z (-i\omega) &= \lim_{\epsilon\to 0}\int_{Z_\epsilon}(-i\omega)\\ &= \lim_{\epsilon\to 0}\int_{Z_\epsilon} d\left(\frac{\nabla s}{s}\right)\\ &= \lim_{\epsilon\to 0}\int_{\partial Z_\epsilon}\frac{\nabla s}{s}.\end{aligned}$$

Note that as ϵ goes to zero, the last integral is supported in an arbitrarily small neighborhood of $\Sigma \cap Z$. Therefore it will suffice to evaluate the limit near each point of $\Sigma \cap Z$. Near such a point, we can assume there are coordinates in some neighborhood $U \subset U_i$ such that

$$s(r_1, \theta_1, r_2, \theta_2, ...) = r_1 e^{i\theta_1}.$$

We then have

$$\frac{\nabla s}{s} = s^{-1}ds + A_i = d\log r_1 + id\theta_1 + A_i.$$

Since

$$\lim_{\epsilon\to 0}\int_{\partial Z_\epsilon \cap U} A_i = 0,$$

we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\partial Z_\epsilon\cap U}\frac{\nabla s}{s} &= \lim_{\epsilon\to 0}\int_{\partial Z_\epsilon\cap U}(d\log r_1 + id\theta_1)\\ &= \pm 2\pi i.\end{aligned}$$

It follows that we have

$$\int_Z (-i\omega) = -2\pi i(Z\cdot\Sigma),$$

as desired ∎

Remark 1.7. *Equivalently, for any β closed $(n-2)$-form,*

$$\frac{1}{2\pi}\int_V \omega\wedge\beta = \int_\Sigma \beta.$$

1.3 From Cohomology to Line Bundles

Theorem 1.8. *Let ω be a closed 2-form on V. If $[\omega/2\pi] \in H^2(V,\mathbb{Z})$, then there exists a line bundle $L \to V$ and a Hermitian connection ∇ on L such that $F = -i\omega$ (here F is the curvature 2-form of ∇).*

Proof. Fix an open covering $V = \cup_{i=1}^N U_i$ and β_i a 1-form on U_i such that $\omega = d\beta_i$ on U_i. We want to find certain $g_{ij} : U_{ij} \to U(1)$ satisfying the cocycle condition. On U_{ij}, $d(\beta_i - \beta_j) = \omega - \omega = 0$, so there exists $f_{ij} : U_{ij} \to \mathbb{R}$ such that $df_{ij} = \beta_i - \beta_j$. On U_{ijk}, we have $d(f_{jk} - f_{ik} + f_{ij}) = 0$, and hence $f_{jk} - f_{ik} + f_{ij} = a_{ijk}$ for some constant a_{ijk}. *Claim:* $[\frac{\omega}{2\pi}] \in H^2(V,\mathbb{Z})$ implies that we can choose the f_{ij}'s such that $a_{ijk} \in \mathbb{Z}$. We then set $g_{ij} = \exp(2\pi i f_{ij})$ and $A_i = -i\beta_i$. Note that the cocycle condtion for g_{ij} follows from the fact that $a_{ijk} \in \mathbb{Z}$. Moreover, we have

$$A_j - A_i = i\beta_i - i\beta_j = idf_{ij} = g_{ij}^{-1} dg_{ij},$$

so the A_i's indeed define a connection on the line bundle defined by the g_{ij}'s. Of course, $dA_i = -id\beta_i = -i\omega$, as desired.

□

1.4 Back to Donaldson's Result

Theorem 1.9. *(Donaldson '96)*
Suppose ω is a symplectic form on a closed manifold V, with $[\omega/(2\pi)] \in H^2(V,\mathbb{Z})$ and $L \to V$ is a line bundle having a connection with curvature $-i\omega$. Then there exists a sequence of sections $s_k \in \Gamma(L^k)$ and constants $C, \delta > 0$ such that for $k >> 0$,

- $\forall x$, $|\bar{\partial} s_k(x)| \le C/\sqrt{k}$
- $\forall x$, $|s_k(x)| \le \delta \Rightarrow |\nabla s_k(x)| \ge \delta$.

Remark 1.10.

- *The connection ∇ on L gives rise to a connection (also denoted by ∇) on L^k with curvature multiplied by k.*
- *The estimates in the theorem are with respect to some fixed compatible almost complex structure J.*
- *In our notation, we have*

$$\bar{\partial} s = \frac{1}{2}(\nabla s + i\nabla s \circ J)$$
$$\partial s = \frac{1}{2}(\nabla s - i\nabla s \circ J).$$

- *In Giroux's notation,* $\nabla' s = \partial s$ *and* $\nabla'' s = \overline{\partial} s$*. Some authors also use* $\nabla^{1,0} s = \partial s$ *and* $\nabla^{0,1} s = \overline{\partial} s$*.*

- *On* V*, we use the metric* $g_k(\cdot, \cdot) := k\omega(\cdot, J\cdot)$ *when discussing properties of the line bundle* L^k*. Note that we're using* pointwise *norms.*

Of fundamental importance is the following algebraic lemma:

Lemma 1.11. *If* $A : \mathbb{C}^n \to \mathbb{C}$ *is* $\mathbb{R}$*-linear and* $||A^{0,1}|| < ||A^{1,0}||$ *(w.r.t the Euclidean metric on* $\mathbb{C}^n$*) then* $\ker A$ *is a codimension* 2 *symplectic subspace.*

Proof. Consider the adjoint map $A^* : \mathbb{C} \to \mathbb{C}^n$. Then

$$(A^{0,1})^* = (A^*)^{0,1} : \mathbb{C} \to \mathbb{C}^n$$
$$(A^{1,0})^* = (A^*)^{1,0} : \mathbb{C} \to \mathbb{C}^n,$$

where the first map is anti $\mathbb{C}$-linear and the second map is $\mathbb{C}$-linear.

Now set

$$v' = (A^{1,0})^*(1), \quad ||(A^{1,0})^*|| = ||v'||$$
$$v'' = (A^{0,1})^*(i), \quad ||(A^{0,1})^*|| = ||v''||$$

By hypothesis, $||v'|| \neq ||v''||$. Now let ω_0 denote the standard symplectic form on $\mathbb{C}^n$, so

$$\begin{aligned}\omega_0(A^*1, A^*i) &= \omega_0(v' + v'', iv' - iv'') \\ &= ||v'||^2 - ||v''||^2 \neq 0\end{aligned}$$

It follows that $\mathrm{span}(A^*1, A^*i)$ is a symplectic subspace of $\mathbb{C}^n$. Then $\mathrm{Ker} A = (\mathrm{Im} A^*)^\perp = i(\mathrm{Im} A^*)^{\omega_0}$ (here the superscript ω_0 denotes the symplectic orthogonal complement). Thus $\mathrm{Im} A^*$ is symplectic, and therefore so is $(\mathrm{Im} A^*)^{\omega_0}$ and hence $\mathrm{Ker} A$. $\square$

Our goal is to build s_k's which are

1. asymptotically holomorphic: $|\overline{\partial} s)k| \leq C/\sqrt{k}$
2. uniformly transverse to 0: $|s| \leq \delta \Rightarrow |\nabla s| \geq \delta$.

The outline of the rest of the talks is roughly:

- Marco: build model sections in a Darboux chart. In the model, ω and the ∇ are standard but J is non-integrable. But as $k \to \infty$, a "zoom effect" kills the effect of the non-integrability. After proving some estimates, we will thus have lots of examples of asymptotically holomorphic s_k's.

- Vincent: Choose among these a uniformly transverse sequence admitting a quantitative version of Sard's theorem.

- Thomas: Prove the quantitative version of Sard's theorem modulo some results about the complexity of semi-algebraic subsets of $\mathbb{C}^n$ (this will involve some complex analysis).

- Sylvain: Discuss the complexity of real semi-algebraic subsets (this will be geometric).

Note: These notes also include talks discussing applications of these ideas, given by Jean-Paul Mohsen and Chris Wendl.

2 Peak Sections - Marco Mazzucchelli

2.1 Building Almost Holomorphic Sections

We now describe the recipe for building almost holomorphic sections of

$$(L^k, \nabla) \to (V, J),$$

where ∇ is a connection with curvature $-ik\omega$. We consider the local setup:

$$(L_0 = \mathbb{C}^n \times \mathbb{C}, |\cdot|, \nabla = d + A) \to (\mathbb{C}^n, \omega_0, J_0),$$

where $|\cdot|$ denotes a Hermitian inner product and $A = \frac{1}{4}\sum_j (z_j d\overline{z}_j - \overline{z}_j dz_j)$, $dA = -i\omega_0$. Consider the section of L_0 given by $f : \mathbb{C}^n \to \mathbb{C}$, $f(z) = \exp(-|z|^2/4)$. We claim that f is holomorphic, i.e. $\nabla^{0,1} f = 0$. Indeed,

$$\begin{aligned}
\nabla^{0,1} f &= d^{0,1} f + A^{0,1} f \\
&= \overline{\partial} f + \frac{1}{4} z d\overline{z} f \\
&= -\frac{1}{4} z d\overline{z} f + \frac{1}{4} z d\overline{z} f \\
&= 0.
\end{aligned}$$

Then

$$\nabla f = \nabla^{1,0} f = \partial f + A^{1,0} f = -\frac{1}{2}\overline{z} dz f,$$

which we view as a section of $(T^*\mathbb{C}^n \otimes L_0, d \otimes \nabla := \widetilde{\nabla})$. Here by definition

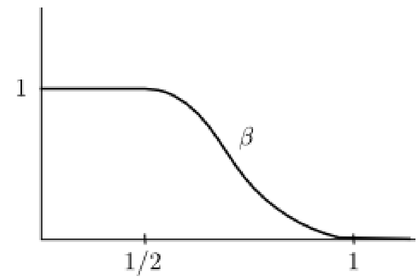
$$\widetilde{\nabla}(\beta \otimes s) = (d\beta) \otimes s + \beta \otimes (\nabla s)$$

for β a section of $T^*\mathbb{C}^n$ and s a section of L_0. For brevity we'll denote $\widetilde{\nabla}$ and its higher derivative cousins again by ∇.

Now for $r \in \mathbb{N}$ we have

$$\begin{aligned}
\nabla^r f &= \nabla^{r-1}(-\frac{1}{2}\overline{z} dz f) \\
&= -\frac{1}{2}\sum_{j=0}^{r-1} \binom{r-1}{j} d^j(\overline{z} dz) \otimes \nabla^{r-1-j} f \\
&= -\frac{1}{2}\overline{z} dz \nabla^{r-1} - \frac{1}{2}(r-1) d\overline{z} \otimes dz \nabla^{r-2} f,
\end{aligned}$$

and therefore $|\nabla^r f| \leq P_r(|z|) f$, where P_r is polynomial of degree r (which might change from line to line).



Now let's make this section compactly supported. Let $\beta : [0,\infty) \to [0,1]$ be a smooth function such that $\beta \equiv 1$ on $[0,1/2]$ and $\beta \equiv 0$ on $[1,\infty)$. For $k \in \mathbb{N}$, let $\beta_k : \mathbb{C}^n \to \mathbb{R}$ be given by

$$\beta_k(z) = \beta(k^{-1/3}|z|^2).$$

Observe that $\beta_k f$ is *not* a holomorphic section, but *almost*:

$$\begin{aligned} |\nabla^r(\beta_k f)| &\leq P_r(|z|)f \\ |\nabla^{0,1}(\beta_k f)| = (\overline{\partial}\beta_k)f + \beta_k \nabla^{0,1} f &= \beta' k^{-1/3} z d\overline{z} f \\ |\nabla^r \nabla^{0,1}(\beta_k f)| &\leq k^{-1/3} P_r(|z|) f. \end{aligned}$$

Now let's plug this local model into our $L^k \to (V,\omega,J,g := \omega(\cdot, J\cdot))$. Let $\phi : (B^{2n}(R),\omega_0) \to (V,\omega)$ be a Darboux chart with $\phi_0 = p \in V$. Without loss of generality we can assume ϕ is J-holomorphic at 0, i.e.

$$d\phi(0) \circ J_0 = J \circ d\phi(0).$$

For any fixed $0 \leq \epsilon < 1$, we can assume (after shrinking R) that

$$(1-\epsilon)|v|_{\mathrm{euc}} \leq |v|_g \leq (1+\epsilon)|v|_{\mathrm{euc}}$$

for any $v \in TB^{2n}(R)$.

Now let $\phi_k(z) := \phi(z/\sqrt{k})$:

$$\begin{tikzcd} (B^{2n}(\sqrt{k}R) \times \mathbb{C}, |\cdot|, d+A) \arrow[r] \arrow[d] & (L^k, h, \nabla) \arrow[d] \\ (B^{2n}(\sqrt{k}R), \omega_0) \arrow[r, "\phi_k"] & (V, k\omega) \end{tikzcd}$$

Note that $\phi_k : (B^{2n}(\sqrt{k}R),\omega_0) \to (V,k\omega)$ is again a symplectomorphism. Here the map $\widetilde{\phi}_k : B^{2n}(\sqrt{k}R) \times \mathbb{C} \to L^k$, $\widetilde{\phi}_k(z,v_0) = (\phi_k(z), v_1)$ is defined as follows. Let $\nabla = d + B$ be the pullback of ∇ to $B^{2n}(\sqrt{k}R)$, and let v_1 be given by parallel transporting v_0 along the radial line from 0 to z in $B^{2n}(\sqrt{k}R)$ using the connection $d+B$.

Exercise 2.1. *$\widetilde{\phi}_k^* B = A$, and therefore ∇ becomes standard under $\widetilde{\phi}_k$.*

Now using $\widetilde{\phi}_k$, we can push forward the section $\beta_k f$ to a compactly supported section of L^k. Note that for any $k \in \mathbb{N}$ and $v \in TB^{2n}(\sqrt{k}R)$ have

$$(1-\epsilon)|v| \le |v|_{kg} \le (1+\epsilon)|v|$$

where $|v|_{kg} = \sqrt{kg((\phi_k)_* v, (\phi_k)_* v)}$. Recall that $J = J_0$ at the origin (pulling back J via ϕ_k) and therefore we have

$$|dJ| \le \frac{\text{const}}{\sqrt{k}}$$
$$|J - J_0| \le \frac{\text{const}}{\sqrt{k}}|z|$$

for constants independent of k.

Now we have

$$\nabla_J^{0,1} f = \frac{1}{2}(\nabla f + i\nabla f J)$$
$$|\nabla^r \nabla_J^{0,1} f| = |\nabla^r(\nabla_J^{0,1} - \nabla^{0,1})f| = |\frac{1}{4}\nabla^r(\overline{z}dz \circ (J - J_0)f)|$$

(recall that $\nabla^{0,1} f = 0$) and therefore

$$|\nabla^r \nabla_J^{0,1} f| \le k^{-1/2} P_r(|z|) f.$$

Similarly, we have

$$|\nabla^r \nabla_J^{0,1} \beta_k f| \le k^{-1/2} P_r(|z|) f.$$

But of course this estimate is still using the Euclidean metric, which we want to replace with the metric $\omega(\cdot, J\cdot)$ on V. Also, we want to replace the connection $d \otimes \nabla$ on $T^*B^{2n} \otimes L$ with $\nabla_{\mathrm{LC}} \otimes \nabla$, where ∇_{LC} is the Levi-Civita connection. Morally, we should replace $P_r(|z|)$ with $P_r(\mathrm{dist}_{kg}(0, z))$.

2.2 More on Peak Sections

Recall the setup: $(L^k, \nabla) \to (V, k\omega, J, kg)$, where L is a line bundle with connection ∇ of curvature $-i\omega$ inducing a connection ∇ on L^k of curvature $-ik\omega$. From last time:

Lemma 2.2. *For any $p \in V$ and $k \in \mathbb{N}$* large*, there is a section $s = s_{p,k}$ such that*

1. *for any $R > 0$, there exists $C_R > 0$ such that for $k >> 0$ we have $|s_{p,k}(q)| \ge C_R$ provided $dist_{kg}(p, q) \le R$*
2. *$|s_{p,k}| \le 1$*
3. *We have*

$$|\nabla^r s(q)| \le P_r(dist_{kg}(p,q))e_k(p,q)$$
$$|\nabla^r \nabla^{0,1} s(q)| \le \frac{1}{\sqrt{k}} P_r(dist_{kg}(p,q))e_k(p,q)$$

where

$$e_k(p,q) = \begin{cases} \exp(-dist_{kg}(p,q)^2/5) & \text{if } dist_{kg}(p,q)^2 \le k^{1/3} \\ 0 & \text{otherwise.} \end{cases}$$

To build candidate almost holomorphic sections, set

$$s = \sum_{p \in \Lambda_k} w_p s_{k,p}$$

where Λ_k is some suitable finite set of points in V, each $w_p \in \mathbb{C}$ with $|w_p| \le 1$.

Lemma 2.3. *Fix $r' \in \mathbb{N}$. Then Λ_k can be chosen (for $k >> 0$) to be $1/\sqrt{k}$-dense (i.e. $\cup_{p \in \Lambda_k} B_g(p, 1/\sqrt{k}) = V$) and such that for any such $\{w_p\}$ and any $0 \le r \le r'$, we have*

$$|\nabla^r s| \le C_r$$
$$|\nabla^r \nabla^{0,1} s| \le C_r/\sqrt{k}.$$

In subsequent lectures we'll show that we can choose $\{w_p\}$ (for $k >> 0$) such that

$$|\nabla^{1,0} s| \ge \epsilon > 0 \quad \text{on} \quad s^{-1}(0),$$

which will imply that s is transverse to the 0-section and there $|\nabla^{0,1} s| < |\nabla^{1,0} s|$.

Proof of Lemma 2.3. For $\rho > 0$, suppose $\Lambda \subset \mathbb{R}$ is discrete with respect to ρ, i.e. $B(x,\rho) \cap B(y,\rho) = \varnothing$ if $x, y \in \Lambda$, $x \ne y$. Then for any $a, b \in \mathbb{N} \setminus \{0\}$, there exists $C_{a,b,\rho} > 0$ such that

$$\sum_{\lambda \in \Lambda} |z - \lambda|^\alpha \exp(-b|z-\lambda|^2) \le C_{a,b,\rho}$$

for any $z \in \mathbb{R}^n$. Here $C_{a,b,\rho}$ is independent of our choice of ρ-discrete Λ.

Now fix a finite atlas for V, $\{\phi_\alpha : U_\alpha \to V^{2n}\}$, with $U''_\alpha \subset\subset U'_\alpha \subset\subset U_\alpha$, such that $\{\phi_\alpha(U''_\alpha)\}$ covers V and

$$\frac{1}{2}|x-y| \le \mathrm{dist}_g(\phi_\alpha(x), \phi_\alpha(y)) \le 2|x-y|.$$

$\square$

Let $\Lambda'_k = \frac{1}{\sqrt{2nk}}(\mathbb{Z}^n \oplus i\mathbb{Z}^n)$, and note that Λ'_k is $\frac{1}{2\sqrt{k}}$-dense in $\mathbb{C}^n$. Set $\Lambda_{k,\alpha} = \phi_\alpha(\Lambda'_k \cap U'_\alpha)$ and $\Lambda_k = \cup_\alpha \Lambda_{k,\alpha}$. Recall that $e_k(x,\lambda) = 0$ if $\mathrm{dist}_g(x,\lambda) > k^{-1/3}$, hence if $x \notin \phi_\alpha(U_\alpha)$. Now for $x \in \phi_\alpha(U_\alpha)$, we have

$$\begin{aligned}
\sum_{\lambda \in \Lambda_{k,\alpha}} \mathrm{dist}_{kg}(x,\lambda)^r e_k(x,\lambda) &\leq \sum_{\lambda \in \Lambda'_k} 2^r k^{r/2} |\phi_\alpha^{-1}(x) - \lambda|^r \exp\left(\frac{-k|\phi_\alpha^{-1}(x) - \lambda|^2}{20}\right) \\
&\leq \sum_{\lambda \in \Lambda'_k} 2^r |k^{1/2}\phi_\alpha^{-1}(x) - \lambda|^r \exp\left(\frac{-k|\phi_\alpha^{-1}(x) - \lambda|^2}{20}\right) \\
&\leq \text{const.}
\end{aligned}$$

3 Quantitative Transversality in Symplectic Geometry - Jean-Paul Mohsen

For $A : V \to W$ a linear map between vector spaces, define

$$\begin{aligned}
\mathrm{Inj}A &= \min_{x \in V,\, ||x||=1} ||Ax|| \\
\mathrm{Surj}A &= \min_{\lambda \in W^*, ||\lambda||=1} ||\lambda \circ A|| = \mathrm{Inj}A^*.
\end{aligned}$$

Observe that A is injective if and only if $\mathrm{Inj}A \neq 0$, and similarly A is surjective if and only if $\mathrm{Surj}A \neq 0$.

The *Transversalization Theorem* will say that we can perturb an approximately holomorphic section of a very positive line bundle L^k to make it transverse to the zero section, with certain estimates. Let (V, ω, J, g) be an almost Kahler manifold, $L \to V$ a complex line bundle with connection ∇^L and curvature $-i\omega$, and $E \to V$ a Hermitian vector bundle with connection ∇^E.

Roughly, we have:

Theorem 3.1. *For s an approximately holomorphic section of $L^k \otimes E$ with $k >> 1$, there exist sections s_1, s_2 of $L^k \otimes E$ such that*

- $s = s_1 + s_2$
- $s_1 \pitchfork 0$ *with estimates*
- s_2 *is small (in some C^r sense).*

Remark 3.2. *For $V_1 \subset V$ compact, we can replace "$s_1 \pitchfork 0$ with estimates" with "$(s_1)_{V_1} \pitchfork 0$ with estimates".*

Theorem 3.3. *For any $\epsilon, C > 0$, $m \in \mathbb{N}$, there exists $\delta > 0$ such that, for all $k >> 1$ and $s \in \Gamma(L^k \otimes E)$ satisfying*

- $||s||_{kg} < C$
- $||\nabla s||_{kg} \leq C$
- $||\nabla'' s||_{kg} \leq C/\sqrt{k}$
- $||\nabla^2 s||_{kg} \leq C$
- $||\nabla(\nabla'' s)||_{kg} \leq C/\sqrt{k}$

there exist $s_1, s_2 \in \Gamma(L^k \otimes E)$ *with* $s = s_1 + s_2$ *such that*

- *For any* $p \in V_1$, $\delta \leq \max\left(||s_1(p)||, Surj_{kg}(\nabla s_1)_{T_pV_1}\right)$
- $||\nabla^i s_2||_{kg} \leq \epsilon$ *for* $0 \leq i \leq m$
- $||\nabla^i \nabla'' s_2||_{kg} \leq \epsilon/\sqrt{k}$ *for* $0 \leq i \leq m-1$.

Proof. The proof of the above theorem involves four steps:
1st step: transversality for real polynomial maps
2nd step: transversality for holomorphic maps
3rd step: local transversality for approximately holomorphic sections
4th step: global transversality for approximately holomorphic sections

We give an outline of the last three steps.

1st step: Let V, W be Hermitian vector spaces, $F : \frac{11}{10}B_V \rightarrow B_W$ a holomorphic map between balls, and $V_1 \subset V$ a real subspace. Fix $\epsilon > 0$. Then we can find $v \in W$ such that

- $||v|| \leq \epsilon$
- for any $x \in V_1 \cap B_V$, $\max\left(||F(x) - v||, \mathrm{Surj}((d_xF)_{V_1})\right) \geq \epsilon/(\log(1/\epsilon))^N$

3rd step: We consider the case where $E = V \times \mathbb{C}^r$ is a trivial bundle over V. Let s be an approximately holomorphic section of $L^k \otimes \mathbb{C}^r$ and let $y_1 \in V_1$. Then there exist sections s_1, s_2 such that

- $s = s_1 + s_2$
- for any $y \in V_1$ with $d_{kg}(y, y_1) \leq 1$, we have $\max\left(||s_1(y)||, \mathrm{Surj}_{kg}(\nabla s_1)_{T_yV_1}\right) \geq \eta_\epsilon$
- $s_2 = v \otimes s_{y_1,k}$, where $v \in \mathbb{C}^r$ with $||v|| \leq \epsilon$ and where $s_{y_1,k}$ is a "peak section" of L^k at y_1
- $\eta_\epsilon = \epsilon/(\log(1/\epsilon))^N$.

4th step: There exists $\Lambda_k = \{y_1, ..., y_{n_k}\} \subset V_1$ such that

- $d_{kg}(y_i, y_j) \geq 1$ for any $i \neq j$
- for any $y \in V_1$, there exists i such that $d_{kg}(y, y_i) \leq 1$

- $\eta_k \leq C k^{\dim V_1/2}$.

Now let $\epsilon_1 \geq \epsilon_2 \geq ... \geq \epsilon_{n_k} > 0$. Again take $E = \mathbb{C}^r$. Consider sections

- $s_2 = \sum_{i=1}^{n_k} v_i \otimes s_{y_i,k}$ for some $v_1, ..., v_{n_k} \in \mathbb{C}^r$ with $||v_i|| \leq \epsilon_i$ for $1 \leq i \leq n_k$
- $s_1 = s - s_2$.

Also, consider sections

- $s_2^j = \sum_{i=1}^{j} v_i \otimes s_{y_i,k}$
- $s_1^j = s - s_2^j$

(so $s_2 = s_2^{n_k}$ and $s_1 = s_1^{n_k}$). By Step 3, we can find v_j such that

- $||v_j|| \leq \epsilon_j$
- for any $y \in V_1$ with $d_{kg}(y, y_j) \leq 1$, we have $\max\left(||s_1^j(y)||, \mathrm{Surj}_{kg}(\nabla^j s_1)_{T_y V_1}\right) \geq \eta_{\epsilon_j}$.

Then we have

$$||s_1(y)|| \geq ||s_1^j(y)|| - \sum_{i=j+1}^{n_k} ||v_i|| \cdot ||s_{y_i,k}(y)||,$$

where

$$\sum_{i=j+1}^{n_k} ||v_i|| \cdot ||s_{y_i,k}(y)|| \leq \sum_{i=j+1}^{n_k} \epsilon_i C \exp\left(-d_{kg}^2(y, y_i)/2\right),$$

and we have a similar estimate for $\mathrm{Surj}_{kg}(\nabla s_1)_{T_y V_1}$. Then

$$\max\left(||s_1(y)||, \mathrm{Surj}_{kg}(\nabla s_1)_{T_y V_1}\right) \geq \eta_{\epsilon_j} - \sum_{i=j+1}^{n_k} \epsilon_i C \exp\left(-d_{kg}^2(y, y_i)/2\right) =: \eta_{\epsilon_j}^*.$$

Question: Can we choose $\epsilon_1 \geq \epsilon_2 \geq ... \geq \epsilon_{n_k} > 0$ such that $\min \eta_j^* \geq \eta > 0$ (where η is independent of k)?
Answer: No, unless we reorder the points of Λ_k!

The idea is to permute the y_i's such that for any $i \neq j$, either $|i - j|$ is "large enough" or else $d_{kg}(y_i, y_j)$ is "large enough".

□

4 Global Theory Modulo Quantitative Sard's Theorem - Vincent Humilière

In this lecture we discuss the global construction of Donaldson hypersurfaces. Recall that Marco showed how to construct a finite subset

$$\Lambda_k = \{p_1, ..., p_{n_k}\} \subset V$$

with the following property. For for any $\underline{w} = (w_1, ..., w_{n_k})$ with $|w_j|$ for $1 \leq j \leq n_k$, let $s_{\underline{w}} = \sum_{j=1}^{n_k} w_j s_{k,p_j}$. Then there exists some $C > 0$ such that for any $k >> 0$ and any such $\underline{w}$, we have

$$|\nabla_J^{0,1} s_{\underline{w}}| \leq C/\sqrt{k}.$$

Our goal now is to prove the following:

Proposition 4.1. *There exists $\epsilon > 0$ such that for any $k >> 0$, there exists $\underline{w}$ such that*

$$|\nabla_J^{1,0} s_{\underline{w}}| > \epsilon \quad on \quad s_{\underline{w}}^{-1}(0).$$

Theorem 4.2. *For any $k >> 0$, there exists a section $s : V \to L^k$ such that*

$$|\nabla^{0,1} s| < |\nabla^{1,0} s| \quad on \quad s^{-1}(0).$$

4.1 Coloring and Strategy

Lemma 4.3. *For any $D > 0$, there exists $N(D) = \mathcal{O}(D^{2n})$ such that, for $k >> 0$, Λ_k can be chosen as before with:*

- *Λ_k is 1-dense with respect to $d_k := d_{kg}$*
- $\sum_{p \in \Lambda_k} d_k(p, \cdot)^r e_k(p, \cdot) \leq C$
- *Λ_k admits a partition $\Lambda_k = I_1^k \cup ... \cup I_{N(D)}^k$ such that for any $p, q \in I_\alpha$, $d(p, q) \geq D$.*

Proof. As before, we have a finite atlas for V, $\{\phi_\beta : U_\beta \to V^{2n}\}$ and open sets $U''_\beta \subset\subset U'_\beta \subset\subset U_\beta$, and Λ_k was constructed such that

$$\phi_\beta(\Lambda_k \cap U'_\beta) = \frac{1}{\sqrt{2nk}}(\mathbb{Z}^n + i\mathbb{Z}^n).$$

Observe that $(\mathbb{Z}^n + i\mathbb{Z}^n)/L(\mathbb{Z}^n + i\mathbb{Z}^n)$, for $L \in \mathbb{N}$, gives a partition of $(\mathbb{Z}^n + i\mathbb{Z}^n)$ such that two elements in the same class are a distance at least L apart. Pushing this forward to V, for L large enough we get a partition of $\Lambda_k \cap U'_\beta$ such that two elements in the same class are a distance at least D apart. We take the union over β of all these partitions. $\square$

Our strategy will be as follows. Fix $D > 0$ and start with an arbitrary $\underline{w}_0$. We inductively adjust the coefficients of $\underline{w}_0$ of color $\alpha \in \{1, ..., N(D)\}$ to get some $\underline{w}_\alpha$. At each step, the change of coefficients

- gives some "controlled transversality" on all d_k 1-balls of color α
- does not kill the controlled transversality previously obtained on balls of color less than α.

More precisely, for any α we will find $\epsilon > 0$ such that

$$|\nabla^{1,0}_J s_{\underline{w}_\alpha}| > \alpha \quad \text{on} \quad s^{-1}_{\underline{w}_\alpha}(0) \cap \cup_{i \in I^k_\beta, \beta \leq \alpha} B_i,$$

where B_i denotes the d_k ball of radius 1 centered at p_i.

4.2 Controlled Transversality

Definition 4.4. *Consider a map $f : U \subset \mathbb{C}^n \to \mathbb{C}$ and a complex number $w \in \mathbb{C}$. We say f is "η-transverse" to w if for any $z \in U$ such that $|f(z) - w| \leq \eta$ we have $|\partial f(z)| \geq \eta$.*

Remark 4.5.
- *If f is holomorphic, f is transverse to w if and only if f is η-transverse to w for some $\eta > 0$.*
- *If f is η-transverse then it is also η' transverse for any $\eta' < \eta$.*
- *If f is η-transverse and $||f - g||_{C^1} \leq \delta < \eta$, then g is $(\eta - \delta)$- transverse.*

We are now almost ready to state our version of the **Quantitative Sard's Lemma**. Let

- $\Delta = B(0, 11/10) \subset \mathbb{C}^n$
- $\Delta^+ = D(0, 22/10) \times ... \times D(0, 22/10) \subset \mathbb{C}^n$
- $Q_p(t) = (-\log t)^{-p} \quad p \in \mathbb{N}, \ \ t > 0$

Theorem 4.6. *(Donaldson) There exists $p \in \mathbb{N}$ such that for any $\delta \in (0, 1/4)$, any $\sigma \leq \delta Q_p(\delta)$ and any $f : \Delta^+ \to \mathbb{C}$ such that $||f||_{C^0} \leq 1$ and $||\overline{\partial} f||_{C^1} \leq \sigma$, there exists $w \in \mathbb{C}$ with $||w|| \leq \delta$ such that f is $\delta Q_p(\delta)$-transverse to w on Δ. Moreover, w can be chosen in any quadrant of $\mathbb{C}$ (here by quadrant we mean any rotation of the standard first quadrant by some angle).*

Remark 4.7.
- *For f holomorphic, if $||f||_{C^0} \leq 1$, there exists w with $||w|| < \delta$ such that f is $\delta Q_p(\delta)$-transverse to w.*
- *For fixed t, $Q_p(t)$ decreases with p (if $t < 1/e$) so if p works, then $p + 1$ also works.*

Recall that for any $p_i \in \Lambda_k$ we have Darboux charts $\phi^k_{p_i}$ which are approximate isometries Let

$$B_i = B_{d_k}(p_i, 1) \subset \phi^k_{p_i}(\Delta) \subset \phi^k_{p_i}(\Delta^+).$$

Then

$$s_{\underline{w}} = (f_i \circ (\phi^k_{p_i})^{-1}) s_{k,p_i}$$

on $\phi^k_{p_i}(\Delta^+)$ defines $f_i^{\underline{w}} : \Delta^+ \to \mathbb{C}$.

Definition 4.8. *We say s is η-transverse if all f_i's are η-transverse to 0.*

4.3 Estimates for the f_i's

Lemma 4.9. *There exists $C > 0$ such that for any $k >> 0$ and any $\underline{w}$, we have*

1. $||f_i^{\underline{w}}||_{C^1(\Delta^+)} \leq C$.
2. $||\overline{\partial} f_i^{\underline{w}}||_{C^1(\Delta^+} \leq C/\sqrt{k}$.
3. *If $||\partial f_i^{\underline{w}}|| > \epsilon$ on $f_i^{-1}(0) \cap \Delta$, then $|\nabla^{1,0} s_{\underline{w}}| > \epsilon/C$ on $s_{\underline{w}}^{-1}(0) \cap B_i$.*

If $\underline{w}'$ coincides with $\underline{w}$ except on I^k_α and such that $|w_i - w'_i| < \delta$, then we have

4. *For any $p_i \in \Lambda_k$, $||f_i^{\underline{w}} - f_i^{\underline{w}'}||_{C^1(\Delta^+)} \leq C\delta$.*
5. *For any $p_i \in \Lambda_k$, if $w_i = w'_i$ then $||f_i^{\underline{w}} - f_i^{\underline{w}'}||_{C^1(\Delta^+)} \leq C\delta \exp(-D^2/5)$.*

Proof idea. There exists $R > 0$, for any $k >> 0$, with $\phi^k_{p_i}(\Delta^+) \subset B_{d_k}(p_i, R)$, and there exists C_R such that $|s_{k,p_i}| \geq C_R > 0$ on $\phi^k_{p_i}(\Delta^+)$ (this was proven in Marco's lecture).

1. We have $s_{\underline{w}} \circ \phi = f_i \cdot (s_{k,p_i} \circ \phi_i)$, with $|f_i| = |s_{\underline{w}} \circ \phi_i| / |s_{k,p_i} \circ \phi_i| \leq C$ for $\phi_i := \phi^k_{p_i} : \Delta^+ \to V$ Then

   $$\nabla s_{\underline{w}} = d(f_i \circ \phi_i^{-1}) \otimes s_{k,p_i} + (f_i \circ \phi_i^{-1}) \nabla s_{k,p_i}$$

   with

   $$||d(f_i \circ \phi_i^{-1}|| \leq |\nabla s_{\underline{w}}| / |s_{k,p_i}| + (|f_i \circ \phi_i^{-1}|) |\nabla s_{k,p_i}| / |s_{k,p_i}| \leq C,$$

   and hence $||df_i|| \leq C$ since ϕ_i is an approximate isometry.

2. We have

   $$\overline{\partial} s_{\underline{w}} = \overline{\partial}(f_i \circ \phi_i^{-1}) \otimes s_{k,p_i} + (f_i \circ \phi_i^{-1}) \overline{\partial} s_{k,p_i}$$
   $$|\overline{\partial}(f_i \circ \phi_i^{-1})| \leq C/\sqrt{k}$$

   hence $|\overline{\partial} f_i| \leq C/\sqrt{k}$.

3. We have $\partial s_{\underline{w}} = \partial(f_i \circ \phi_i^{-1}) \otimes s_{k,p_i} + (f_i \circ \phi_i^{-1})\partial s_{k,p_i}$ with $f_i \circ \phi_i^{-1} = 0$ on $s_{\underline{w}}^{-1}(0)$, so
$$|\partial s_{\underline{w}}| = |s_{k,p_i}||\partial(f_i \circ \phi_i^{-1})| > (C_R)(\epsilon).$$
4. We have $s_{\underline{w}-\underline{w}'} = (f_i^{\underline{w}} - f_i^{\underline{w}'})s_{k,p_i}$ hence since $|underlinew - \underline{w}'| \leq \delta$, by (1) we have $||f_i^{\underline{w}} - f_i^{\underline{w}'}|| \leq C\delta$.
5. For any $p_j \in I_\alpha^k \setminus \{p_i\}$ with $d_k(p_j, p_i) \geq D$, we have
$$||s_{k,p_j}|| \leq C \exp(-D^2/5),$$
and therefore
$$\begin{aligned} ||s_{\underline{w}-\underline{w}'}|| &\leq C\delta \exp(-D^2/5) \\ ||f_i^{\underline{w}} - f_i^{\underline{w}'}|| &\leq C\delta \exp(-D^2/5). \end{aligned}$$

□

4.4 Induction on Colors

The present goal is to inductively construct a sequence $\underline{w_\alpha}$ such that for any α there exists $\eta_\alpha > 0$ such that $s_{\underline{w_\alpha}}$ is η_α-transverse on
$$V_\alpha = \cup_{i \in I_\beta^k,\, \beta \leq \alpha} B_i.$$

Proposition 4.10. *There exists $0 < \rho < 1$ and $p \in \mathbb{N}$ such that if $s_{\underline{w_\alpha}}$ is η_α-transverse on V_α with $\eta_\alpha < \rho$, and if*

1. $1/\sqrt{k} \leq \eta_\alpha Q_p(\eta_\alpha)$
2. $\exp(-D^2/5) \leq Q_p(\eta_\alpha)$,

then there exists $\underline{w_{\alpha+1}}$ such that $s_{\underline{w_{\alpha+1}}}$ is $\eta_{\alpha+1}$-transverse (on $V_{\alpha+1}$) with $\eta_{\alpha+1} = \eta_\alpha Q_p(\eta_\alpha)$ (recall that $Q_p(t) = (-\log t)^{-p}$).

Proof. We have $f_i^\alpha : \Delta^+ \to \mathbb{C}$ with
$$\begin{aligned} ||f_i^\alpha||_{C^0(\Delta^+)} &\leq C \\ ||\overline{\partial} f_i^\alpha||_{C^1(\Delta^+)} &\leq C/\sqrt{k}. \end{aligned}$$

Applying Sard's theorem to $\frac{1}{C} f_i^\alpha$ for $i \in I_{\alpha+1}^k$, there exists p_0 such that for any δ, k with
$$\delta \in (0, 1/4) \tag{1}$$
$$1/\sqrt{k} \leq \delta Q_{p_0}(\delta), \tag{2}$$

there exists $v_i \in \mathbb{C}$ with $|v_i| < \delta$, where $C^{-1} f_i^\alpha$ is $\delta Q_{p_0}(\delta)$-transverse to v_i on Δ.

Set $w_{\alpha+1,j} = \begin{cases} w_{\alpha+1,j} - Cv_j & \text{if } j \in I^k_{\alpha+1} \\ w_{\alpha,j} & \text{otherwise} \end{cases}$

If

$$|c\delta| < 1, \tag{3}$$

we can use the quadrant condition to ensure that $w_{\alpha,j} - Cv_j$ actually lies in the unit disk.

Now we need estimates on $s_{\underline{w_{\alpha+1}}}$ on each B_i, $i \in I_\beta$, $\beta \leq \alpha + 1$. Let $i \in I_\beta$ for $\beta \leq \alpha$. Then $s_{\underline{w_\alpha}}$ is η_α-transverse (on V_α) and for any j, $|w_{\alpha+1,j} - w_{\alpha,j}| \leq C\delta$. By the fourth part of the lemma, we have

$$||f_i^{\alpha+1} - f_i^\alpha||_{C^1(\Delta^+)} \leq C^2\delta,$$

hence $s_{\underline{w_{\alpha+1}}}$ is $\eta_\alpha - C^2\delta$ transverse. Note that this is relevent only if $C^2\delta < \eta_\alpha$.

Now let $i \in I_{\alpha+1}$. Introduce an auxiliary $\underline{w}'$ defined by

$$w'_j = \begin{cases} w_{\alpha,j} & \text{if } j \neq i \\ w_{\alpha,i} - Cv_i & \text{otherwise} \end{cases}$$

Compare $s_{\underline{w_\alpha}}$ and $s_{\underline{w}'}$:

$$s_{\underline{w_\alpha} - \underline{w}'} = Cv_i s_{k,p_i},$$

so $f'_i - f_i^\alpha = -Cv_i$, hence

$$\begin{aligned} &f_i^\alpha \text{ is } C\delta Q_p(\delta) - \text{transverse to } Cv_i \\ &f'_i \text{ is } C\delta Q_{p_0}(\delta) - \text{transverse to } 0. \end{aligned}$$

Now compare $s_{\underline{w}'}$ with $s_{\underline{w_{\alpha+1}}}$. Observe that $\underline{w}'$ and $\underline{w_{\alpha+1}}$ coincide except on $I_{\alpha+1} \setminus \{i\}$. By the fifth part of the lemma, we get

$$||f_i^{\alpha+1} - f'_i||_{C^1(\Delta^+)} \leq C\delta \exp(-D^2/5)$$

and so $f_i^{\alpha+1}$ is $C\delta Q_{p_0}(\delta) - C\delta\exp(-D^2/5)$-transverse to 0. This is relevant only if

$$\exp(-D^2/5) \leq Q_{p_0}(\delta). \tag{4}$$

Now we choose δ (and the other parameters). Let ρ be small enough that

$$\frac{\eta_\alpha}{2C^2} < \min(1/2, 1/C)$$

and let $\delta = \frac{\eta_\alpha}{2C^2}$. Then (1),(2), and (4) are satisfied.

Now we consider $p = p_0 + 1$. Since $Q_p(t) \to 0$ as $t \to 0$, for ρ small enough we have $Q_{p_0}(\delta) >> Q_p(\eta_\alpha)$. Then

$$\begin{aligned} &(A) \quad 1/\sqrt{k} \leq \eta_\alpha Q_p(\eta_\alpha) \Longrightarrow (2) \quad (\text{since } 1/\sqrt{k} \leq \delta Q_{p_0}(\delta)) \\ &(B) \quad \exp(-D^2/5) \leq Q_p(\eta_\alpha) \Longrightarrow Q_{p_0}(\delta) >> \exp(-D^2/5). \end{aligned}$$

Since $C\delta Q_{p_0}(\delta) - C\delta \exp(-D^2/5) \approx C\delta Q_{p_0}(\delta) > \eta_\alpha Q_p(\eta_\alpha) = \eta_{\alpha+1}$, $f_i^{\alpha+1}$ is $\eta_{\alpha+1}$-transverse for and $i \in I_{\alpha+1}$. Using condition (2), for any $i \in I_\beta$ with $\beta \leq \alpha$, $f_i^{\alpha+1}$ is $(\eta_\alpha - C^2\delta)$-transverse. Here $\eta_\alpha - C^2\delta = \eta_\alpha(1 - 1/2) = (1/2)\eta_\alpha > \eta_\alpha Q_p(\eta_\alpha)$ for ρ small enough because $Q_p(t) \to 0$ as $t \to 0$. □

Now that we have proven the proposition, we need to show that we can apply it repeatedly, each time getting conditions (A) and (B).

Exercise 4.11. *Let $p > 0$, $(\eta_\alpha)_{\alpha\in\mathbb{N}^*}$, $\eta_{\alpha+1} = \eta_\alpha + p\log(\eta_\alpha)$. Then for any $q > p$, there exists $\beta \in \mathbb{N}$ such that $\eta_\alpha < q(\alpha+\beta)\log(\alpha+\beta)$.*

Assuming the exercise, let $\eta_0 = \rho$ (as given by the proposition) and set $\eta_{\alpha+1} = \eta_\alpha Q_p(\eta_\alpha)$. Applying the exercise to $-\log(\eta_\alpha)$, we get

$$\begin{aligned} Q_p(\eta_\alpha) = \eta_\alpha^{-p} \geq \frac{1}{(q(\alpha+\beta)\log(\alpha+\beta))^p} &\geq \frac{C}{(\alpha\log\alpha)^p} \\ &\geq \frac{C}{(N(D)\log N(D))^p} \\ &\geq \frac{1}{D^{2np+1}} \\ &\geq \exp(-D^2/5) \end{aligned}$$

$(N(D) = \mathcal{O}(D^{2n}))$. So (B) is satisfied at any step for D large enough. Condition (A) $(1/\sqrt{k} \leq \eta_\alpha Q_p(\eta_\alpha))$ is satisfied for $k >> 0$ for any α (recall that there are a finite number of colors).

5 Quantitative Sard's Theorem Modulo Yomdim's Results - Thomas Letendre

First some notation:

$$\begin{aligned} &\Delta = \{z \in \mathbb{C}^n \mid |z| \leq 11/10\} \\ &\Delta^+ = \{z \in \mathbb{C}^n \mid |z_j| \leq 22/10 \; \forall \; j\}. \end{aligned}$$

For $\sigma > 0$ let

$$H_\sigma = \{f : \Delta^+ \to \mathbb{C} \text{ smooth} \mid ||f||_{C^0(\Delta^+)} \leq 1, \; ||\overline{\partial} f||_{C^1(\Delta^+)} \leq \sigma\}.$$

For $p \in \mathbb{N}$ and $\eta > 0$,

$$Q_p(\eta) = \left(\frac{1}{\log(1/\eta)}\right)^p.$$

Note that for $\eta \leq 1/4$, $Q_p(\eta) \leq \left(\frac{1}{\log(4)}\right)^p$.

Definition 5.1. *A smooth function $f : U \to \mathbb{C}$ is called η-transverse to $w \in \mathbb{C}$ over U if for any $z \in U$ such that $|f(z) - w| \leq \eta$, we have $|\partial_z f| \geq \eta$.*

Theorem 5.2. *(Donaldson) There is some $p \in \mathbb{N}$ depending only on the dimension n such that for any $\eta \in (0, 1/4)$, $\sigma \in (0, \eta Q_p(\eta))$, and $f \in H_\sigma$, there exists $w \in \mathbb{C}$ with $|w| \leq \eta$ such that f is $\eta Q_p(\eta)$-tranverse to w over Δ. Moreover, we can assume $Re(w), Im(w) > 0$ (in fact we can pick w is any quadrant).*

Here is the outline:

1. Approximate holomorphic functions by polynomials
2. Prove the theorem for holomorphic functions
3. Prove the general case (modulo Hormander's methods)
4. Hormander's methods

1) We begin with

Lemma 5.3. *Let $f : \Delta^+ \to \mathbb{C}$ be a holomorphic function such that $||f||_{C^0(\Delta^+)} \leq 1$. There exists $C > 0$ such that for any $0 < \epsilon \leq 1/2$, there exists a polynomial g of degree at most $C \log(\epsilon^{-1})$ such that $||f - g||_{C^1(\Delta)} \leq \epsilon$.*

Proof. Let $\Gamma = \{z \in \mathbb{C}^n \mid |z_j| = 22/10 \ \forall \ j\}$. For any $z \in \Delta$, Cauchy's formula gives

$$f(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(w)}{(w_1 - z_1)...(w_n - z_n)} dw_1...dw_n$$
$$f(z) = \sum a_{i_1...i_n} z_1^{i_1}...z_n^{i_n}$$

with

$$a_{i_1...i_n} = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(w)}{w_1^{i_1+1}...w_n^{i_n+1}} dw.$$

For $s \in \mathbb{N}$, let $g_s = \sum_{i_j \leq s} a_{i_1...i_n} z_1^{i_1}...z_n^{i_n}$. For $z \in \Delta$, we have

$$\begin{aligned} f(z) - g_s(z) &= \sum_{\exists j \text{ st } i_j > s} a_{i_1...i_n} z_1^{i_1}...z_n^{i_n} \\ &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(w)}{w_1...w_n} \sum_{\exists j \text{ st } i_j > s} \left(\frac{z_1}{w_1}\right)^{i_i} ... \left(\frac{z_n}{w_n}\right)^{i_n} dw \\ &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(w)}{w_1...w_n} E_z(w) dw, \end{aligned}$$

where $E_z(w) := \sum_{\exists j \text{ st } i_j > s} \left(\frac{z_1}{w_1}\right)^{i_i} ... \left(\frac{z_n}{w_n}\right)^{i_n}$. We have

$$\begin{aligned}
|f(z) - g_s(z)| &\leq ||f||_{C^0(\Delta^+)} ||E_z||_{C^0(\Gamma)} \leq ||E_z||_{C^0(\Gamma)} \\
|E_z(w)| &\leq \sum_{i_1 ... i_n \text{ st } \exists j \text{ st } i_j > s} \frac{1}{2^{i_1}} ... \frac{1}{2^{i_n}} \\
&\leq n \left(\sum_{i_1 \in \mathbb{N}} \frac{1}{2^{i_1}}\right) ... \left(\sum_{i_n > s} \frac{1}{2^{i_n}}\right) \\
&\leq n 2^{n-s-1},
\end{aligned}$$

and therefore

$$||f - g_s||_{C^0(\Delta)} \leq n 2^{n-s-1}.$$

Similarly,

$$\begin{aligned}
\left|\frac{\partial}{\partial z_j}(f - g_s)(z)\right| &= \left|\frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(w)}{w_1 ... w_n} \frac{\partial}{\partial z_j}(E_z(w)) dw\right| \\
&\leq \left|\left|\frac{\partial}{\partial z_j}(E_z)\right|\right|_{C^0(\Gamma)} \leq (s + n + 1) 2^{n-s-1} \\
||\partial(f - g_s)||^2_{C^0(\Delta)} &= \sum_j \left|\left|\frac{\partial}{\partial z_j}(f - g_s)\right|\right|^2_{C^0(\Delta)}
\end{aligned}$$

so

$$||\partial(f - g_s)||_{C^0(\Delta)} \leq \sqrt{n}(s + n + 1) 2^{n-s-1}.$$

Then for some C, λ, we have $||f - g_s||_{C^1(\Delta)} \leq Ce^{-\lambda s}$.

Now let $0 < \epsilon \leq 1/2$. Observe that $Ce^{-\lambda s} \leq \epsilon$ is equivalent to $s \geq \frac{\log(C) + \log(\epsilon^{-1})}{\lambda}$. Define $g := g_s$ for $s = \left\lfloor \frac{\log(C) + \log(\epsilon^{-1})}{\lambda} \right\rfloor + 1$. Then g is a polynomial with $\deg(s) \leq ns$. Note that $\deg(g) \leq n \left(\frac{\log(C) + \log(\epsilon^{-1})}{\lambda} + 1\right) \leq C' \log(\epsilon^{-1})$. □

2) Now we prove the theorem for holomorphic functions. Let $f : \Delta^+ \to \mathbb{C}$ be a holomorphic function such that $||f||_{C^0(\Delta^+)} \leq 1$ and $0 < \epsilon < 1/4$. Let

$$S^f = \{z \in \Delta \mid |\partial_z f| \leq \epsilon\}.$$

Note that f is ϵ-transverse to $w \in \mathbb{C}$ over Δ if and only if $w \in N_{f,\epsilon}$ (the ϵ-neighborhood of $f(S^f)$). Let g be the polynomial given by the lemma, so $d = \deg(g) \leq C \log(\epsilon^{-1})$ and $||f - g||_{C^1(\Delta)} \leq \epsilon$. Let

$$S^g = \{z \in \Delta \mid |\partial_z g| \leq 2\epsilon\}$$

and let $N_{g,\epsilon}$ be the ϵ-neighborhood of $g(S^g)$. Then $S^f \subset S^g$ (since $||f-g||_{C^1(\Delta)} \leq \epsilon$) and $f(S^f) \subset f(S^g) \subset N_{g,\epsilon}$ (since $||f-g||_{C^0(\Delta)} \leq \epsilon$) and thus $N_{f,\epsilon} \subset N_{g,2\epsilon}$.

Complexity of semi-algebraic sets:

Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and let

$$S_p = \{x \in \mathbb{R}^n \mid ||x|| \leq 1,\ P(x) \leq 1\}$$

and for $\theta > 0$,

$$S_p(\theta) = \{x \in \mathbb{R}^n \mid ||x|| \leq 1,\ |P(x)| \leq 1+\theta\}.$$

Theorem 5.4. *(Yomdin, Gromov, Donaldson, Mohsen)*
There exists constants C, V depending only on n such that for any P, there exists arbitarily small $\theta > 0$ such that S may decomposed into A pieces:

$$S_P = S_1 \cup \ldots \cup S_A$$

and any two points in the same S_j can be joined by a path of length at most L in $S_p(\theta)$ with $A, L \leq Cd^V$ where $d = \deg(P)$.

Proof. Set

$$S^g = \{z \in \Delta \mid \left|\frac{\partial_z g}{2\epsilon}\right|^2 \leq 1\} = S_p.$$

Take θ as given by the theorem: $A, L \leq C\deg(P)^V = C(2(d-1))^V$ and $S^g = S_1 \cup \ldots \cup S_A$. For any $z_1, z_2 \in S_j$, $|g(z_1) - g(z_2)| \leq 2\epsilon L$ and z_1, z_2 can be joined by a path of length at most L in $S_p(\theta)$. Moreover, $g(S^g)$ can be covered by A disks of radius at most $2\epsilon L$.

Now $N_{g,2\epsilon}$ is contained in a union of A disks of radius at most $2\epsilon(L+1)$, hence its area is at most $A\pi(2\epsilon(L+1))^2$. Let

$$\Omega_p = \{w \in \mathbb{C} \mid |w| \leq \rho \text{ and } \mathrm{Re}(w) > 0,\ \mathrm{Im}(w) > 0\}.$$

If $\frac{1}{4}\pi\rho^2 < A\pi(2\epsilon(L+1))^2$, i.e. $\rho > \sqrt{A}(4\epsilon(L+1))$, there is $w \in \Omega_p \setminus N_{g,2\epsilon}$ such that f is ϵ-tranverse to w over Δ. Choose $\rho_0 = 4\sqrt{A}\epsilon(L+1) + \epsilon$. Then there exists

$$w \in \Omega := \{w \in \mathbb{C} \mid \mathrm{Re}(w) > 0,\ \mathrm{Im}(w) > 0\}$$

such that $|w| \leq \rho_0$ and f is ϵ-tranverse to w over Δ.

Since $A, L \leq C(2(d-1))^v$, we have $\rho \leq \epsilon P(d)$ for P a polynomial and $d \leq C' \log(\epsilon^{-1})$. Thus

$$\begin{aligned}\rho_0 &\leq \epsilon \tilde{P}(\log(\epsilon^{-1})) \\ &\leq C''\epsilon \log(\epsilon^{-1})^p\end{aligned}$$

for some $p \in \mathbb{N}$ and $\tilde{P}$ a polynomial. Here C'' and P depend only on n. $\square$

Let $h_p(\epsilon) = C\epsilon \log(\epsilon^{-1})^p$.

Exercise 5.5. *$h_p(\epsilon) \to 0$ as $\epsilon \to 0$, and $h_p : (0, \epsilon^{-p}) \to (0, h_p(\epsilon^{-p}))$ is strictly increasing (hence invertible).*

Up to increasing p, $h_p(\epsilon^{-p}) \geq 1/4$. Let $0 < \eta < 1/4$, $\epsilon = h_p^{-1}(\eta)$. There exists $w \in \Omega$ with $|w| \leq h_p(\epsilon) = \eta$ and ϵ-transverse to w. We have

$$\eta Q_p(\eta) = C\epsilon \left(\frac{log(\epsilon^{-1})}{\log(\epsilon^{-1}) - \log(C) - \log(\log(\epsilon^{-1}))} \right)^p \leq 2\epsilon C$$

if $\eta \leq \eta_0$.

Then for $\eta \leq \eta_0$, $\frac{\eta Q_p(\eta)}{2C} \leq \epsilon$ so f is $\frac{\eta Q_p(\eta)}{2C}$-tranverse to w. Increasing p again, we can

- lift the condition $\eta \leq \eta_0$
- erase $1/(2C)$.

For all $0 < \eta \leq 1/4$, there exists $w \in \Omega$ such that (for the new p) $|w| \leq \eta$, f is $\eta Q_p(\eta)$-tranvserse to w over Δ.

3) <u>The General Case</u>:

Let $0 < \eta < 1/4$, $\sigma \geq 0$, $f \in H_\sigma$, and $3/4 < r' < 1$.

Theorem 5.6. *(Hormander) For any (smooth) $(0,1)$-form g over Δ^+ such that $\overline{\partial} g = 0$, there exists $u : \Delta^+ \to C$ (smooth) such that $\overline{\partial} u = g$ and $||u||_{L^2(r'\Delta^+)} \leq K||g||_{L^2(\Delta^+)}$, with K depending only on r'.*

Applying this to $g = \overline{\partial} f$, we get a smooth function u with $\overline{\partial} u = \overline{\partial} f$ and $||u||_{L^2(r'\Delta^+)} \leq K||\overline{\partial} f||_{L^2(\Delta^+)}$ Let $\widehat{f} := f - u$ and note that $\widehat{f}$ is holomorphic. Let $3/4 < r < r'$ and $\epsilon = (r' - r)/2$. Let $B_\epsilon(z)$ be the ball in $\mathbb{C}^n$ with center z and radius ϵ. We have the following analytic lemma (we omit the proof):

Lemma 5.7. *For any $z \in r\Delta^+$, $|u(z)| \leq C(||u||_{L^2(B_\epsilon(z))} + ||\overline{\partial} f||_{C^0(B_\epsilon(z))})$, with C depending only on ϵ.*

Therefore we have

$$\begin{aligned} ||u||_{C^0(r\Delta^+)} &\leq C(||u||_{L^2(r'\Delta^+)} + ||\overline{\partial} f||_{C^0(\Delta^+)}) \\ &\leq C'||\overline{\partial} f||_{C^0(\Delta^+)} \end{aligned}$$

(since $||u||_{L^2(r'\Delta^+)} \leq K||\overline{\partial} f||_{L^2(\Delta^+)}$). By a similar computation, we have

$$||du||_{C^0(r\Delta^+)} \leq C(||\overline{\partial} f||_{C^0(\Delta^+)} + ||d\overline{\partial} f||_{C^0(\Delta^+)})$$

hence

$$\begin{aligned}||\widehat{f}-f||_{C^1(r\Delta^+)} &= ||u||_{C^1(r\Delta+)}\\ &\leq C(||\overline{\partial}f||_{C^0(\Delta^+)}+||d\overline{\partial}f||_{C^0(\Delta^+)})\\ &\leq C\sigma\end{aligned}$$

(recall that $f \in H_\sigma$). Set $\eta' = \frac{\eta}{1+C\sigma}$.

Now there exists $w \in \Omega$ such that $|w| \leq \eta'$ and $\frac{\widehat{f}}{1+C\sigma}$ is $\eta' Q_p(\eta')$-tranverse to w over Δ. Let $w' := (1+C\sigma)w$, $w \in \Omega$, $|w'| \leq \eta$ and $\widehat{f}$ is $(1+C\sigma)\eta' Q_p(\eta')$-transverse to w' over Δ. Note that $(1+C\sigma)\eta' Q_p(\eta') \geq \eta Q_{p'}(\eta)$ for some $p' > p$.

So up to increasing p, $\widehat{f}$ is $\eta Q_p(\eta)$-transverse to w' over Δ. Since $||f-\widehat{f}||_{C^1(r\Delta^+)} \leq C\sigma$, we have that f is $(\eta Q_p(\eta) - C\sigma)$-tranverse to w' over Δ. If $\sigma \leq \frac{1}{2C}\eta Q_p(\eta)$, f is $\frac{\eta Q_p(\eta)}{2}$-tranverse to w over Δ. Increasing p again, f is $\eta Q_p(\eta)$-tranverse to w over Δ and this is true for any $\sigma \leq \eta Q_p(\eta)$.

4) Hormander's L^2 methods:

Let $\phi : \Delta^+ \to \mathbb{R}$ be continuous, and let

$$\begin{aligned}L^2(\phi) &= \{f : \Delta^+ \to \mathbb{C} \mid \int_{\Delta^+} |f|^2 e^{-\phi} < \infty\}\\ L^2_{(0,q)}(\phi) &= \{(0,q)-\text{forms on } \Delta^+ \text{ with coefficients in } L^2(\phi)\}.\end{aligned}$$

We write a typical element of the latter set as $\omega = \sum \omega_I \overline{dz_I}$.

Hilbert spaces: Define an inner product on $L^2_{(0,q)}(\phi)$ by $\langle w, \eta\rangle_\phi = \sum_{|I|=q}\sum_{\Delta^+} \omega_I \overline{\eta}_I e^{-\phi}$. Fix $\phi_1, \phi_2, \phi_3 : \Delta^+ \to \mathbb{C}$ continuous functions. Then $\overline{\partial}$ defines a closed, densely defined operator $T : L^2(\phi_1) \to L^2_{(0,1)}(\phi_2)$. Let

$$D_T = \{u \in L^2(\phi) \mid \overline{\partial}u \in L^2_{(0,1)}(\phi_2)\}.$$

For any $u \in D_T$, $Tu = \overline{\partial}u$. Then D_T is dense because $C_c^\infty(\Delta^+) \subset D_T$. T is closed (has a closed graph) because $\overline{\partial}$ is continuous in the distribution sense: $u_n \to u \implies \overline{\partial}u_n \to \overline{\partial}u$. We cna define $T^* : L^2_{(0,1)}(\phi_2) \to L^2(\phi_1)$. Let

$$D_{T^*} = \{v \in L^2_{(0,1)}(\phi_2) \mid \exists C_v \text{ such that } \forall u \in D_T,\ |\langle v, Tu\rangle_{\phi_1}| \leq C_V||u||_{\phi_1}\}.$$

For $v \in D_{T^*}$, one can extend $\langle T\cdot, v\rangle_{\phi_2}$ continuously to $L^2(\phi_1)$, hence there exists $T^*v \in L^2(\phi_1)$ such that for any $v \in D_T$, $\langle Tu, v\rangle_{\phi_2} = \langle u, T^*v\rangle_{\phi_1}$.

Facts:

- For any $u \in D_T$, $v \in D_{T^*}$, we have $\langle Tu, v\rangle_{\phi_2} = \langle u, T^*v\rangle_{\phi_1}$.
- T^* is closed.
- D_{T^*} is dense in $L^2_{(0,1)}(\phi_2)$.

- $\mathrm{Im}(T)^{\perp} \subset \mathrm{Ker}(T^*)$.

Proposition 5.8. *Let F be closed subspace of $L^2_{(0,1)}(\phi_2)$ such that $Im(T) \subset F$. Then $F = Im(T)$ if and only if for any $f \in F \cap D_{T^*}$ we have*

$$||f||_{\phi_2} \leq C(||T^*f||_{\phi_1}).$$

Proof. If $F = \mathrm{Im}(T)$, $B = \{f \in F \cap D_{T^*} \mid ||T^*f||_{\phi_1} \leq 1\}$, for any $v \in F$ take u such that $Tu = v$. For any $f \in B$, we have

$$|\langle v, f\rangle| = |\langle Tu, f\rangle| = |\langle u, T^*f\rangle| \leq ||u||_{\phi_1}.$$

Then for any $v \in F$, $\sup_{f\in B} |\langle v, f\rangle| < \infty$. Thus $\sup_{f\in B} ||\langle \cdot, f\rangle|| < \infty$ (by Banach-Steinhaus). Since $||\langle \cdot, f\rangle|| = ||f||_{\phi_2}$, B must be bounded by some C, so for any $f \in D_{T^*} \cap F$, $||f||_{\phi_2} \leq C||T^*f||_{\phi_1}$.

Conversely, assume that for any $f \in F \cap D_{T^*}$, we have $||f||_{\phi_2} \leq C||T^*f||_{\phi_1}$. Let $g \in F$. We claim that for any $f \in D_{T^*}$, $|\langle g, f\rangle| \leq C||g||_{\phi_2}||T^*f||_{\phi_1}$ if $f \in F^{\perp} \subset (\mathrm{Im}T)^{\perp} \subset \mathrm{Ker}T^*$. If $f \in F \cap D_{T^*}$, we have

$$|\langle g, f\rangle| \leq ||g||_{\phi_2}||f||_{\phi_2} \leq C||g||_{\phi_2}||T^*f||_{\phi_1}.$$

We have a well-defined Lipschitz map $T^*f \mapsto \langle g, f\rangle_{\phi_2}$, $\mathrm{Im}(T^*) \to \mathbb{C}$. Extend this to $\beta : L^2(\phi_1) \to \mathbb{C}$ with $||\beta|| \leq C||g||_{\phi_2}$ (this is possible by Hahn-Banach). Take $u \in L^2(\phi_1)$ such that $\beta = \langle \cdot, u\rangle$. Then $||u||_{\phi_1} = ||\beta|| \leq C||g||_{\phi_2}$ and for any $f \in D_{T^*}$, $\langle u, T^*f\rangle = \langle g, f\rangle$. For any $f \in D_{T^{**}} = D_T$ and $f \in D_{T^*}$, $\langle Tu, f\rangle = \langle g, f\rangle$. Since D_{T^*} is dense, this implies that $g = Tu$ and $||u|| \leq C||g||$. $\square$

Now define $S : L^2_{(0,1)}(\phi_2) \to L^2_{(0,2)}(\phi_3)$ as $S = \overline{\partial}$. To prove Hormander's theorem, we need to show:

$$\exists\, c \text{ such that } \forall f \in D_{T^*} \cap \mathrm{Ker}S,\ ||f||_{\phi_2} \leq C||T^*f||_{\phi_1}$$
$$\exists\, c \text{ such that } \forall f \in D_{T^*} \cap D_S,\ ||f||_{\phi_2} \leq C(||T^*f||\phi_1 + ||Sf||_{\phi_3}).$$

<u>Fact</u> (without proof):
There exist $C > 0$, $\phi_1, \phi_2, \phi_3 : \Delta^+ \to \mathbb{R}$ (smooth) such that

1. $0 = \phi_1 = \phi_2 = \phi_3$ on $r'\Delta^+$
2. $\phi_3 \geq \phi_2 \geq \phi_1$
3. $||f||^2_{\phi_2} \leq C^2(||T^*f||^2_{\phi_1} + ||Sf||^2_{\phi_3})$.

Applying the proposition: for any $g \in L^2_{(0,1)}(\phi_2)$ such that $\overline{\partial}g = 0$, there exists $u \in L^2(\phi_1)$ such that $\overline{\partial}u = g$ and $||u||_{\phi_1} \leq C||g||_{\phi_2}$.
<u>Fact</u>: If g is smooth then u is also smooth.

Then $g \in W^s$ implies that $u \in W^{s+1}$, and g smooth implies that $u \in W^s$ for all s, so by Sobolev's lemma u is in fact smooth.

Finally, we want a bound for n:

$$||u||^2_{L^2(r'\Delta^+)} = \int_{r'\Delta^+} |u|^2 = \int_{r'\Delta^+} |u|^2 e^{-\phi_1} \leq C^2 ||g||^2_{\phi_2} \leq K||g||^2_{L^2(\Delta^+)}.$$

6 Quantitative Transversality in Symplectic Geometry II - Jean-Paul Mohsen

We discuss applications of Donaldson's techniques to

1. symplectic manifolds
2. symplectic submanifolds and real hypersurfaces
3. contact manifolds (Ibort, Martinez, Presas)
4. symplectic isotopies (Auroux).

Let V, W be Hermitian vector spaces and $A : V \to W$ a $\mathbb{C}$-linear map. For A just linear over $\mathbb{R}$, we can write $A = A' + A''$. Then $||A''|| < \text{Surj}A$ implies that $\text{Ker}A$ is an "approximately complex subspace".

Recall that

$$\text{Surj}A = \min_{||\lambda||=1,\ \lambda\ \mathbb{R}-\text{linear functional on } W} ||\lambda \circ A||.$$

Proposition 6.1. *For any $\epsilon > 0$, there exists $\eta > 0$ such that $||A''|| < \eta SurjA$ implies that, for any $v \in KerA$ with $||v|| = 1$, there exists $w \in KerA$ with $d(iv, w) < \epsilon$.*

Proof. Let

$$E = \{\mu \in V^* \text{ with } \mu|_{\text{Ker}A} = 0\} = \{\mu = \lambda \circ A \mid \lambda \in W^*\}.$$

For $v \in \text{Ker}A$, we have

$$A(iv) = iA'v - iA''v = -2iA''v.$$

Then

$$\begin{aligned}
|\mu(iv)| &= |\lambda(A(iv))| = 2|\lambda(iA''v)| \\
&\leq 2||\lambda||\ ||A''|| \leq 2\eta||\lambda||\text{Surj}A \\
&\leq 2\eta||\lambda \circ A|| \\
&= \epsilon||\mu|| \quad \text{for} \quad \epsilon = 2\eta.
\end{aligned}$$

But then

$$\min_{w \in \mathrm{Ker}A} d(iv, w) = ||v_2|| = \max_{\mu \in E,\, ||\mu||=1} |\mu(iv)|,$$

where $iv = (v_1, v_2) \in \mathrm{Ker}A \oplus \mathrm{Ker\ A}^{\perp}$. □

We have the following corollary:

Theorem 6.2. *Let s be a section of the Hermitian vector bundle $L^k \otimes E$, and let $\Sigma = s^{-1}(0)$. Then*

$$||\nabla'' s|| << \mathit{Surj}\, \nabla s \Longrightarrow \Sigma \ \text{ is a symplectic submanifold.}$$

Now let $H \subset V$ be a *real* hyperplane. Recall that the Levi *complex* hyperplane is given by $H \cap iH$.

Proposition 6.3. 1. *If A is $\mathbb{C}$-linear, we have $\mathit{Surj}\ A|_H = \mathit{Surj}\ A|_{H \cap iH}$.*

2. *If A is $\mathbb{R}$-linear, then $\mathit{Surj}\ A|_H - 2||A''|| \leq \mathit{Surj}\ A|_{H \cap iH} \leq \mathit{Surj}\ A|_H$.*

Proof. 1. Let $\lambda : W \to \mathbb{R}$ with $||\lambda|| = 1$ such that $||\lambda \circ A_{H \cap iH}|| = \mathrm{Surj} A_{H \cap iH}$. Write $H = \mathbb{R}x + H \cap iH$. Observe that there exists θ such that $\lambda(e^{i\theta} Ax) = 0$. Then let $\lambda_\theta(w) = \lambda(e^{i\theta} w)$. We have

$$\begin{aligned}
\mathrm{Surj} A_{H \cap iH} &\leq \mathrm{Surj} A_H \ (\text{since } H \cap iH \subset H) \\
&\leq ||\lambda_\theta \circ A|| \\
&= ||\lambda_\theta \circ A_{H \cap iH}|| \\
&= ||\lambda \circ A_{H \cap iH}|| \ (\text{since } A \ \text{ is } \mathbb{C}-\text{linear}) \\
&= \mathrm{Surj}\ A_{H \cap iH}.
\end{aligned}$$

2. We have

$$\begin{aligned}
\mathrm{Surj}\ A_{H \cap iH} &\geq \mathrm{Surj}\ A'_{H \cap iH} - ||A''|| \ (\text{by the Lipschitz property of Surj}) \\
&= \mathrm{Surj}\ A'|_H - ||A''|| \\
&\geq \mathrm{Surj}\ A_H - 2||A''||.
\end{aligned}$$

□

Theorem 6.4. *Let s_0 be an approximately holomorphic section of $L^k \otimes E$ and V_1 a submanifold. Then there exists $s \approx s_0$ such that for any $p \in V_1$, we have*

$$\eta \leq \max(||s||, \mathit{Surj}_{g_k}(\nabla s)_{TV_1}).$$

Theorem 6.5. *Assume now that V_1 is a real hypersurface. With the same s_0 and s as above, we also have*

$$\eta/2 \leq \max(||s||, \mathit{Surj}_{g_k}(\nabla s)_{TV_1 \cap iTV_1}).$$

Proof. We have

$$\begin{aligned}\mathrm{Surj}_{g_k}(\nabla s)_{TV_1\cap iTV_1} &\geq \mathrm{Surj}_{g_k}(\nabla s)_{TV_1} - 2\|\nabla'' s\| \\ &\geq \mathrm{Surj}(\nabla s)_{TV_1} - 2C/\sqrt{k} \text{ (since s is approximately holomorphic)} \\ &\geq \mathrm{Surj}(\nabla s)_{TV_1} - \eta/2 \quad (\text{since } k >> 1).\end{aligned}$$

□

Contact theory:

Let $(V, \omega = d\alpha_V)$ be an exact symplectic manifold and let $V_1 \subset V$ be a real hypersurface. We call V_1 contact if α_V restricts to a contact form on V_1. We assert that there exists an ω-compatible almost complex structure J on V such that $\xi = \mathrm{Ker}\alpha$ is a complex subspace.

Theorem 6.6. *Let s_V be an appromxiately holomorphic section of $L^k \otimes E$ and let s denote its restriction to V_1. Let $\Sigma_1 = s^{-1}(0) \subset V_1$. Suppose that for any $p \in \Sigma_1$, $\|\nabla'' s_V\| <<$ Surj$(\nabla(s))_\xi$ (which can be achieved by the above). Then Σ_1 is a contact submanifold (in V_1) of codimension $2rank_{\mathbb{C}}(E)$.*

Isotopy properties:

Consider the following data:

- (X, ω_X), (Y, ω_Y), $(X \times Y, \omega_X \oplus \omega_Y)$
- L_X complex line bundle with curvature $-i\omega_X$
- L_Y complex line bundle with curvature $-i\omega_Y$
- E_X Hermitian vector bundle
- $L = L_X \otimes L_Y$ complex line bundle with curvature $-i(\omega_X \oplus \omega_Y)$
- s_X section of $L_X^k \otimes E_X$
- s_Y section of L_Y^k
- $s_X \otimes s_Y$ section of $L^k \otimes E_x$.

Theorem 6.7. *With the usual symplectic data V, ω, J, g, L, let s_1, s_2 be approximately holomorphic sections of $L^k \otimes E$ which are η-transverse to 0. Then there exists an isotopy $(s_t)_{t\in[1,2]}$ such that for any t:*

- *s_t is approximately holomorphic*
- *s_t is η_1-tranverse to 0.*

Proof. Let $V_2 = V \times \mathbb{C}$ and $V_3 = V \times [1,2]$. Then $TV_3 = TV \times \mathbb{R}$ and the Levi direcitons are given by $TV \otimes \{0\}$. Let $s = s_1 \otimes s_{k,1} + s_2 \otimes s_{k,2}$. The transversality theorem gives $\sigma \approx s$, where σ is η-transverse to 0. Along $p \in V_3$, we have

$$\max(||\sigma||, \mathrm{Surj}_{g_k}(\nabla\sigma)_{TV\otimes\{0\}}) \geq \eta_1,$$

so we get a family

$$s_1 \longrightarrow \sigma_1 \longrightarrow \sigma_t \longrightarrow \sigma_2 \longrightarrow s_2,$$

where the extrapolation between s_1 and σ_1 can just be taken as $(1-t)s_1 + t\sigma_1$, again by the Lipschitz property of Surj. $\square$

7 Yomdim's Theory - Sylvain Courte

Let $P \in \mathcal{P}_d^* = \{P : B^m \to \mathbb{R}$ polynomial of deg $\leq d$, $|$ 1 is a regular value of P and $P|_{S^{m-1}}\}$. Let $\Sigma = \{P = 1\}$. Our goal is to bound the complexity of σ in terms of d. Here by complexity we mean:

- the number of connected components (c.c.)
- the diameter of connected components in the "path-length" metric.

Theorem 7.1. *(Yomdin, Donaldson, Gromov) There are constants C and ν (depending only on m) such that for any $P \in \mathcal{P}_d^*$ and $\Sigma = \{P = 1\}$, we have*

- *$\#$c.c.$(\Sigma) \leq Cd^\nu$*
- *diam(c.c.$(\Sigma)) \leq Cd^\nu$.*

Notation: We will call a quantity assigned to $P \in \mathcal{P}_d^*$ *p-bounded* if it satisfies such a bound as above. Also, a set is *p-bounded* is $\#$*c.c.* and diam(*c.c.*) are p-bounded.

Remark 7.2. *1. If a set if covered by a p-bounded number of (connected) sets of p-bounded diameter, then it is p-bounded.*

2. $\Omega = \{P \leq 1\}$ is also p-bounded.

Proof. Let $\{\Sigma_i\}$ denote the connected components of Σ, and let $\Omega_i = \{x \in \Omega \mid d_\Omega(x, \Sigma_i) \leq 2\}$. We claim that $\Omega = \cup\Omega_i$ and $\mathrm{diam}(\Omega_i) \leq 2 + \mathrm{diam}\Sigma_i + 2$, so Ω is p-bounded by 1). $\square$

Application of the theorem: Let $\epsilon > 0$ and let g be a complex polynomial of degree $\approx -\ln\epsilon$, with $g : B^{2n} \subset \mathbb{C}^n \to \mathbb{C}$. Then for $P = \frac{|\partial g|^2}{\epsilon^2}$ (a real polynomial) we consider

$$\Omega = \{P \leq 1\} = \text{the set of } \epsilon - \text{critical points}$$
$$g(\Omega) = \text{the set of } \epsilon - \text{critical values}$$

and we have

$$\#c.c.(\Omega) \le C(-\ln \epsilon)^\nu$$
$$\operatorname{diam} c.c.(\Omega) \le C(-\ln \epsilon)^\nu.$$

By the mean value theorem, we have

$$\operatorname{area}(g(\Omega)) \le C(-\ln \epsilon)^\nu C(-\ln \epsilon)^\nu \epsilon^2 \to 0 \quad \text{as } \epsilon \to 0.$$

Proof of theorem. We use induction on m.
$\underline{m=1}$: Σ consists of a most d points.

$\underline{m=2}$: Let $E \subset [-1,1]$ be the set of exceptional values for t, where we view P as a function of z and t:

$$E = \{P = 1,\ z^2 + t^2 = 1\} \cup \{P = 1,\ \frac{\partial P}{\partial z} = 0\}.$$

By Bezout's theorem, we have $|E| \le 2d + d(d-1)$. So we've covered Σ by a p-bounded number of sets. As for their diameters, we will use Crofton's formula:

Theorem 7.3. *For C a curve in $\mathbb{R}^m$, we have*

$$\int_{AGr(m-1,m)} \#(C \cap P) dP = K \cdot \mathit{length}(C),$$

where $AGr(m-1,m)$ denotes the affine Grassmannian of hyperplanes.

The application of Crofton's formula is as follows. For C an algebraic curve of degree d in B^m, for almost every P we have $\#(C \cap P) \le d$, so $\operatorname{length}(C) \le d$. Thus for $m = 2$, Crofton's formula implies that $C = \Sigma$ has p-bounded length.

$\underline{m-1 \Longrightarrow m}$:
Let $\mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-2} \times \mathbb{R}$, with respective coordinates t, y_i, z, where we think of z as the "height function". Let $\pi : \Sigma \to [-1,1]$ be the projection onto the t coordinate and let $\Sigma_t = \pi^{-1}(t)$. Let

$$C = \{P = 1,\ \frac{\partial P}{\partial y_i} = 0\} = \cup_t \operatorname{Crit}(z|_{\Sigma_t})$$

(note that this is a "curve").

We will need to arrange some general position conditions:

1. $(\pm 1, 0, 0) \notin \Sigma$

2. C is a smooth curve intersecting S^{m-1} transversally

3. for all but finitely many t, Σ_t is smooth and tranverse to S^{m-1}

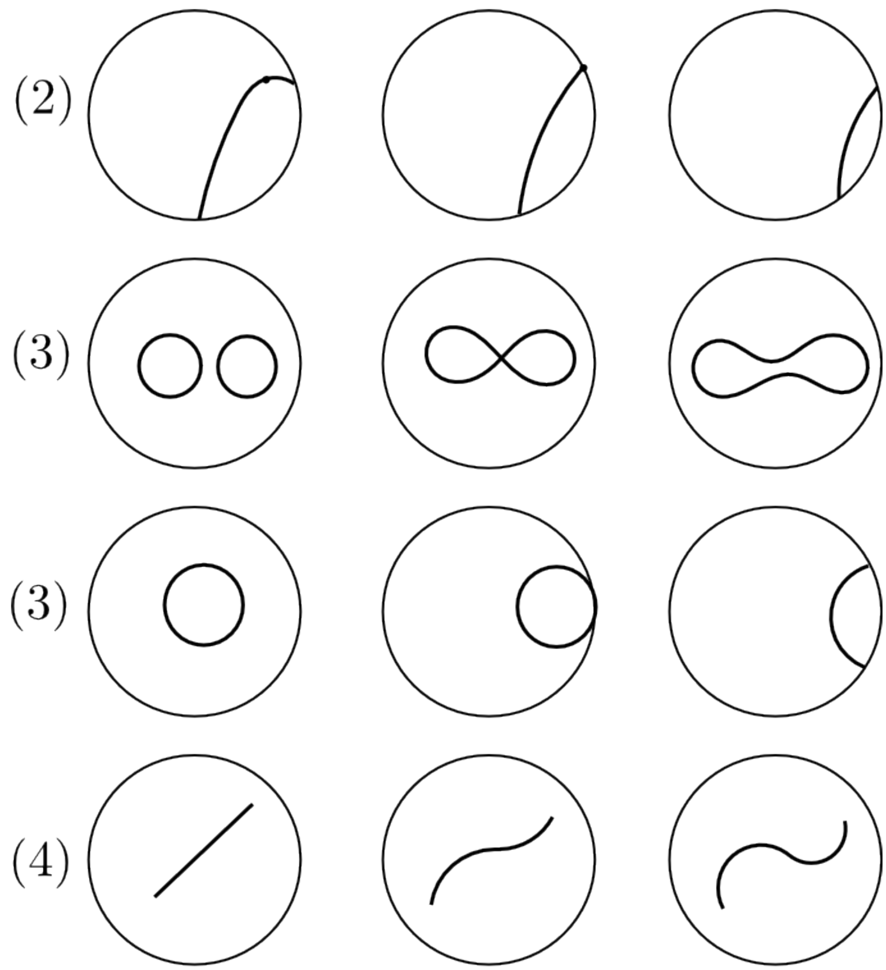


Figure 1: Depiction of the accidents (for $m = 3$).

4. for all but finitely many t, $z|_{\Sigma_t}$ is Morse.

Let $E = \{\text{"accidental parameters" } t\} \cup \{\pm 1\}$.

Lemma 7.4. *For any d, there exists an open and dense subset $\mathcal{P}_d^{**} \subset \mathcal{P}_d^*$ such that (1),(2),(3),(4) are satisfied and E is p-bounded.*

Restricting to $P \in \mathcal{P}_d^{**}$, $[-1,1] \setminus E = \cup J_\beta$ is a union of open intervals. Let $\Sigma_\beta = \pi^{-1}(J_B)$. By induction, Σ_β has a p-bounded number of connected components.

As for diameter, obesrve that there are two kinds of components of Σ_β. Let $\{\Sigma_\beta^i\}$ be the connected components. The either:
i) Σ_β^i does not meet S^{m-1}. Let $x_1, x_2 \in \Sigma_\beta^i$ correspond to t_1 and t_2. Then $d(x_1,x_2) \leq \mathrm{diam}(\Sigma_\beta^i, t_1) + \mathrm{length}(C) + \mathrm{diam}(\Sigma_\beta^i, t_2)$. The first and third terms are p-bounded by induction, while the second term is p-bounded by Crofton's formula.

ii) Σ_β^i touches S^{m-1}. Then similarly, $\Sigma \cap S^{m-1}$ is p-bounded by induction.

Now let $\Sigma^* = \cup \Sigma_\beta = \Sigma \setminus \pi^{-1}(E)$. Then Σ^* is covered by a p-bounded number of sets of p-bounded diameter. Σ^* is dense in Σ, so $\Sigma = \cup_{i,\beta} \overline{\Sigma_\beta^i}$, hence $\mathrm{diam}\overline{\Sigma_\beta^i}$ is p-bounded, which implies the result. □

Now to tie the remaining loose ends, how do we go from $\mathcal{P}_d^{**}$ to $\mathcal{P}_d^*$? We have that for any $P \in \mathcal{P}_d^{**}$, $\#c.c.(\Sigma) \leq Cd^\nu$ and $\mathrm{diam}(c.c.(\Sigma)) \leq Cd^\nu$. We claim:

1. $\#c.c.(\Sigma)$ is locally constant on $\mathcal{P}_d^*$ (this is an artifact of the conditions on $*$)

2. $\mathrm{diam} c.c.(\Sigma)$ is smooth in $\mathcal{P}_d^*$.

<u>Crofton's formula</u>:

We claim that $Gr(m-k,m)$ has a unique $O(m)$-invariant probability measure. We can then use the fibration $AGr(m-k,m) \to Gr(m-k,m)$ with fiber $\mathbb{R}^k$ to get a measure on $AGr(m-k,m)$.

Now we have

Theorem 7.5. *There exists $c(k,m)$ such that for any X^k submanifold of $\mathbb{R}^m$, we have*

$$\int_{Agr(m-k,m)} \#(X \cap P)dP = c(k,m)vol_k(X).$$

Proof. We have

$$\begin{aligned}\int_{Agr(m-k,m)} \#(X\cap P)dP &= \int_{P\in Gr(m-k,m)}\int_{v\in P^\perp} \#(X\cap(P+v))dvdP \\ &= \int_P\int_v \#\pi_p^{-1}(v)dvdP \qquad (\pi_p : X\to P^\perp) \\ &= \int_P\int_X |\text{Jac } d\pi_p(x)|dxdP \\ &= \int_X\int_P |\text{Jac } d\pi_p(x)|dPdx \\ &= c(k,m)\int_X dx \\ &= c(k,m)\text{vol}(X),\end{aligned}$$

where we have used the change of variables formula, Fubini's theorem, and we have noted that $\int_P |\text{Jac} d\pi_p(x)|dP$ is independent of x. □

8 Transversality in Gromov-Witten Theory - Chris Wendl

References:

- Cieliebak-Mohnke '07 - genus 0 case
- Ionel-Parker '13
- A. Gerstenberg, A. Krestienchine - PhD theses to appear
- The MathSciNet review of Cieliebak-Mohnke by Usher.

8.1 The Problem (in Genus 0)

To a symplectic manifold (V^{2n},ω) we want to associate Gromov-Witten invariants $GW^{(V,\omega)}_{0,m,A}$: $H^*(V,\mathbb{Q})^{\otimes m}\to\mathbb{Q}$ for $m\geq 0$ and $A\in H_2(V)$, which requires an associated almost complex structure J which is ω-tame. Morally, this invariant counts "the number of J-holomorphic spheres $u : S^2\to V$ homologous to A with m marked points $z_1,...,z_m\in S^2$ such that for $j=1,...,m$, there exists $u(z_j)$ a submanifold Poincare dual to α_j. Here $GW^{(V,\omega)}_{0,m,A}(\alpha_1,...,\alpha_n)$ "equals"

$$\int_{\overline{\mathcal{M}}^A_{0,m}(V,J)} \text{ev}_1^*\alpha_1\cup...\cup\text{ev}_m^*\alpha_m,$$

where

$$\begin{aligned}\mathcal{M}^A_{0,m}(V,J) = \{(u : S^2\to V,\ \underline{z}=(z_1,...,z_m)\in(S^2)^m \text{ (distinct points) } \mid \\ \overline{\partial}u=0,\ [u]=A\}/\text{biholomorphic reparametrization.}\end{aligned}$$

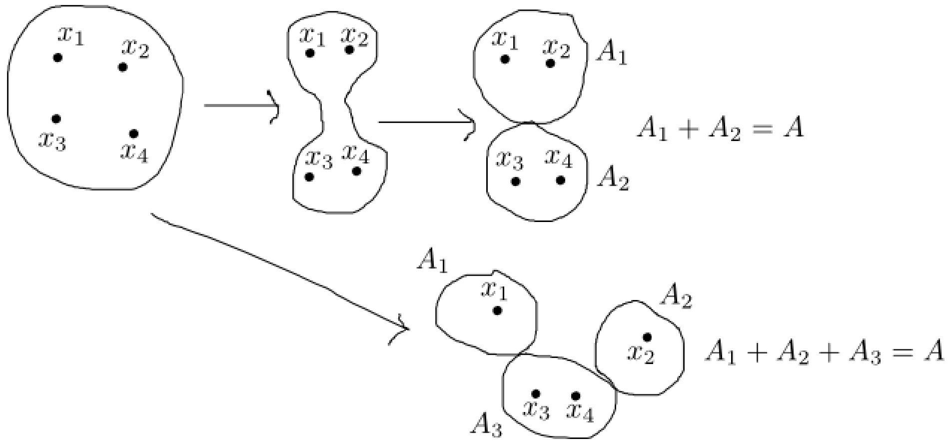


Figure 2: In order for the last degeneration to be stable, the homology classes must satisfy $A_1, A_2 \neq 0$.

Then for $j = 1, ..., m$ we have evaluation maps $\mathrm{ev}_j : \mathcal{M}^A_{0,m}(V, J) \to V$. Using the Fredholme index, we can compute the virtual dimension over $\mathbb{R}$ of $\mathcal{M}^A_{0,m}(V, J)$ to be $2(n-3) + 2c_1(A) + 2m$.

Here $\overline{\mathcal{M}^A_{0,m}(V, J)}$ is the set of stable nodal J-holomorphic curves, where "stable" means that for each constant component, we have

$$\#\text{ marked points} + \#\text{ nodes} \geq 3.$$

Setting $\partial\overline{\mathcal{M}} := \overline{\mathcal{M}} \setminus \mathcal{M}$, we have

$$\partial\overline{\mathcal{M}} = \cup \text{ strata with virtual dimension} \leq \text{ vir. dim } \mathcal{M} - 2.$$

If (*) all moduli spaces are smooth (manifolds or orbifolds) of dim = virtual dim (really we want the linearized Cauchy-Riemann operators to be surjective) then $\mathrm{ev} : \overline{\mathcal{M}}^A_{0,m}(V, J) \to V^m$ is a (rational) "pseudocycle") (c.f. McDuff-Salamon).

<u>The Problem</u>: (*) is almost never satisfied...

Perturbing J generically makes $\mathcal{M}$ smooth <u>only</u> near <u>simple</u> curves (i.e. not multiply covered); it fails if there is symmetry.

8.2 Part of the Solution ("If You're Not Part of the Solution You're Part of the Problem")

If $m \geq 3$, then

$$\mathcal{M}^A_{0,m}(V, J) \cong \{(u : S^2 \to V,\ (0, 1, \infty, z_4, ..., z_m) \mid \overline{\partial}_J u = 0,\ [u] = A\}.$$

<u>Idea</u>: replace $J(p \in V)$ with $J(z \in S^2, p \in V)$ (generic). Then

$$du(z) + J(z, u(z)) \circ du(z) \circ i = 0,$$

hence $\mathcal{M}^A_{(0,m)}(V,J)$ is a smooth manifold of dim = vir. dim. This helps with the multiple covering transversality, but there's still a drawback: when bubbling occurs, a bubble may correspond to a single point in the domain but could still be multipy covered, in which case transversality still fails.

Idea (Ruan / McDuff-Salamon): Assume (V,ω) is *semipositive*, i.e. for any $A \in \pi_2(V)$, if $\omega(A) > 0$ and $c_1(A) \geq 3-n$, then $c_1(A) \geq 0$. Now $codim \partial \overline{\mathcal{M}} \geq 2$, hence the Gromov Witten invariants are actualy $\mathbb{Z}$-valued. Note that this condition does not rule out multiple-covering in bubbles, but they are ruled out for index reasons.

8.3 A Fantasy of a Solution (for the Non-Semipositive Case)

We have a forgetful map

$$\pi : \overline{\mathcal{M}}^A_{0,m}(V,J) \to \widehat{\mathcal{M}}_{0,m} = \{\text{nodal Riemann sphere with m marked points}\}.$$

Idea: Let J depend on points $p = u(z) \in V$, $\pi(u) =: \Sigma_z \in \widehat{\mathcal{M}}_{0,k}$.
Problems:

1. When $m = 0$, $\widehat{\mathcal{M}}_{0,0} = \{\text{pt}\}$, so $\widehat{\mathcal{M}}_{0,0}$ doesn't "know" about bubbling.
2. Even for stable maps, we are forced to consider unstable domains, and unfortunately the moduli space of unstable Riemann spheres with its natural topology is not even Hausdorff...

Recap: We want J to depend on points in $\overline{\mathcal{M}}_{0,m}$ (stable nodal Riemann spheres) and ensure that only stable domains appear.

8.4 Making Fantasy Reality

For $[\omega] \in H^2(V;\mathbb{Z})$, let $W_k \subset V$ be a Donaldson hypersurface of degree $k \in \mathbb{N}$. For a fixed J (ω-compatible), we can assume W_k is "almost J-holomorphic" for $k >> 0$, i.e. there exists J' C^0-close to J such that W_k is J'-holomorphic (J' may be just ω-tame).

Then $u \in \mathcal{M}_{0,m}(V,J')$ implies that $[u]\cdot[W_k] = k\omega(u) =: l \geq k$. Unless u is contained in W_k, generically it intersects W_k at l points, hence it has $l!$ lifts to an element of

$$\mathcal{M}_{0,m+l}(V,J',W_k) := \{u \in \mathcal{M}_{0,m+l}(V,J') \mid \text{for the last l marked points } z_l,\dots,z_{l+m},\ u(z_l),\dots,u(z_{l+m}) \in W_k\}.$$

Idea: This will force domains to have at least 3 marked points!

Lemma 8.1. *Given an ω-compatible J and an almost J-holomorphic hypersurface W_k for $k >> 1$, J has a C^0-small neighborhppd $U_J \subset \{C^\infty\ \omega - \text{tame a.c.s.}\}$ such that*

1. *There exists $k_* > 0$ depending only on (V,ω,J) such that for any $J' \in U_J$, all J'-holomorphic spheres $u : S^2 \to V$ satisfy $c_1(u) \leq k_*\omega(u)$*

2. *for any $k >> 1$, $U_{J,W_k} = \{J' \in U_J \mid W_k$ is $J' -$ holomorphic$\}$ is non-empty and connected.*

Lemma 8.2. *If k is sufficiently large and $J' \in U_{J,W_k}$ generic, then there are no J'-holomorphic spheres contained in W_k.*

Proof. Assume $u : S^2 \to W_k$ is without loss of generality simple. Its index as a curve in W_k is

$$0 \leq \mathrm{ind}(u) = 2(n-4) + 2\langle c_1(TW_k), [u]\rangle$$

where $c_1(TW_k) = c_1(TV|_{W_k}) - c_1(N_{W_k}) = c_1(TV|_{W_k}) - k[\omega|_{W_k}]$, hence

$$\begin{aligned} \mathrm{ind}(u) &= 2(n-4) + 2c_1(u) - k\omega(u) \\ &\leq 2(n-4) - 2(k-k_*)\omega(u) \\ &\leq 2(n-4) - 2(k-k_*) < 0 \end{aligned}$$

if $k >> 1$. □

Lemma 8.3. *If k is sufficiently large and $J' \in U_{J,W_k}$ generic, then every nonconstant J'-holomorphic $u : S^2 \to V$ intersects W_k in at least 3 distinct points of its domain.*

Proof. Let $u : S^2 \to V$ be without loss of generality simple. Let $u^{-1}(W_k) = \{z_1, ..., z_N\}$, where the local intersection index at z_j is $l_j \in \mathbb{N}$. Then $\sum_{j=1}^N l_j = [u] \cdot [W_k] = k\omega(u)$. Now u belongs to the moduli space of curves with N marked points intersecting W_k with these conditions, and so

$$\begin{aligned} 0 \leq \text{vir. dim.} &= 2(n-3) + 2c_1(u) + 2N - 2k\omega(u) \\ &\leq 2(n-3) + 2(k_* - k) + 2N, \end{aligned}$$

hence if $k >> 0$, then also $N >> 0$. □